where $\lambda = 2^{\min(1, \varepsilon)} > 1$. This is sufficient to ensure convergence of (3.26), for any $\xi \in \mathbb{R}$. It immediately follows from (3.28) that the convergence is uniform on compact sets.

Remark. While being more restrictive than (3.1), the condition (3.27) is still very mild. In practice one requires much stronger decay for the $h(n)$. For filter construction purposes, one even restricts oneself to the case where only finitely many $h(n)$ are different from zero.

It is however not sufficient to know that $\hat{\eta}_\infty$ is well defined. In order to avoid situations such as depicted in Figure 4, we require that (i) $\hat{\eta}_\infty$ has sufficient decay, so that η_∞ is sufficiently regular (at least continuous), and (ii) η_l converges to η_∞, pointwise, for $l \to \infty$.

To ensure the decay, for $|\xi| \to \infty$, of $\hat{\eta}_\infty(\xi)$, we shall use the same trick as in subsection 2B, i.e., we shall require that $m_0(\xi)$ is divisible by $(1 + e^{i\xi})^N$, for some $N > 0$. The precise statement is given in the following lemma, using an estimation technique of P. Tchamitchian [5].

LEMMA 3.2. *If $m_0(\xi) = (1 + e^{i\xi})]^N \mathscr{F}(\xi)$, where $\mathscr{F}(\xi) = \sum_n f(n) e^{in\xi}$ satisfies*

$$\sum |f(n)|\,|n|^\varepsilon < \infty \quad \textit{for some} \quad \varepsilon > 0 \tag{3.29}$$

and

$$\sup_{\xi \in \mathbb{R}} |\mathscr{F}(\xi)| = B, \tag{3.30}$$

then there exists $C > 0$ such that, for all $\xi \in \mathbb{R}$,

$$\left| \prod_{j=1}^{\infty} m_0(2^{-j}\xi) \right| \le C(1 + |\xi|)^{-N + (\log B)/(\log 2)}. \tag{3.31}$$

Remarks. 1. It follows from (3.31) that η_∞ is continuous if H satisfies all the above conditions, and if $B < 2^{N-1}$.

2. The condition (3.29) will automatically be satisfied if

$$\sum_n |h(n)|\,|n|^{N+\varepsilon} < \infty. \tag{3.32}$$

Proof: Since $\prod_{j=1}^{\infty}\cos(2^{-j}x) = x^{-1}\sin x$, we have

$$\prod_{j=1}^{\infty} m_0(2^{-j}\xi) = \left[e^{i\xi/2}\prod_{j=1}^{\infty}\cos(2^{-j-1}\xi)\right]^N \prod_{j=1}^{\infty}\mathscr{F}(2^{-j}\xi)$$

(3.33)

$$= e^{iN\xi/2}\left(\frac{\sin\frac{1}{2}\xi}{\frac{1}{2}\xi}\right)^N \prod_{j=1}^{\infty}\mathscr{F}(2^{-j}\xi),$$

where the right-hand side converges uniformly on compact sets because of (3.29). There exists therefore a constant C such that, for all $|\xi| \leq 1$,

$$\left|\prod_{j=1}^{\infty} m_0(2^{-j}\xi)\right| \leq C. \tag{3.34}$$

Take now $|\xi| > 1$. Determine $j_0 \in \mathbf{N}$ such that

$$2^{-j_0}|\xi| < 1 \leq 2^{-j_0+1}|\xi|,$$

i.e.,

$$\log|\xi|/\log 2 < j_0 \leq 1 + \log|\xi|/\log 2.$$

Then

$$\left|\prod_{j=1}^{\infty}\mathscr{F}(2^{-j}\xi)\right| = \left|\prod_{j=1}^{j_0}\mathscr{F}(2^{-j}\xi)\right| \cdot \left|\prod_{j=1}^{\infty}\mathscr{F}(2^{-j}2^{-j_0}\xi)\right|$$

(3.35)

$$\leq B^{j_0}\prod_{j=1}^{\infty}\left(1 + 2^{-j\epsilon}\sum_n |f(n)|\,|n|^{\epsilon}\right)$$

$$\leq C\exp\{\log B \cdot \log|\xi|/\log 2\},$$

To estimate $\prod_{j=1}^{\infty}\mathscr{F}(2^{-j}2^{-j_0}\xi)$ we have used the same argument as in the proof of Lemma 3.1. This is allowed since $\Sigma f(n) = \mathscr{F}(0) = m_0(0) = 1$. Together, (3.35), (3.34) and (3.33) imply (3.31).

In our search for "regularity" we have, so far, only used one of the special conditions on the $h(n)$, derived in subsection 3A, namely (3.4), $\Sigma_n h(n) = \sqrt{2}$. And even that has not played a critical role, since it was only used for normalization purposes, and we could have as easily normalized by any other constant which happened to be the sum of the $h(n)$. For our last step, the proof that the histograms η_l converge pointwise to the continuous function η_∞ (assuming B is not too large), we need an extra ingredient, namely $|m_0(\xi)| \leq 1$. Since, however (see (3.20)), as a consequence of (3.2)–(3.3), $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$, this condition is automatically fulfilled for $h(n)$ satisfying (3.18)–(3.19).

PROPOSITION 3.3. *Define* $m_0(\xi) = 2^{-1/2}\sum_n h(n)e^{in\xi}$, *where the* $h(n)$ *satisfy* (3.18), (3.19). *Suppose moreover that*

$$m_0(\xi) = \left[\tfrac{1}{2}(1+e^{i\xi})\right]^N \mathscr{F}(\xi), \tag{3.36}$$

with $\mathscr{F}(\xi) = \sum_n f(n)e^{in\xi}$ *such that*

$$\sum_n |f(n)|\,|n|^{\varepsilon} < \infty \quad \text{for some} \quad \varepsilon > 0 \tag{3.37}$$

and

$$\sup_{\xi\in\mathbf{R}} |\mathscr{F}(\xi)| = B < 2^{N-1}. \tag{3.38}$$

Then the piecewise constant functions η_l, *defined recursively by*

$$\eta_l(x) = \sqrt{2}\sum_n h(n)\eta_{l-1}(2x-n), \tag{3.39}$$

with

$$\eta_0(x) = \chi_{]-1/2,1/2]}(x),$$

converge pointwise to the continuous function η_∞ *defined by*

$$\hat{\eta}_\infty(\xi) = (2\pi)^{-1/2}\prod_{j=1}^{\infty} m_0(2^{-j}\xi).$$

Proof: 1. As an intermediate step, we prove $\mu_l \to \eta_\infty$, pointwise, where the μ_l are defined in the same recursive way as the η_l, but starting from a different initial function,

$$\mu_0(x) = \begin{cases} 1+x, & -1 \le x \le 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

2. Taking Fourier transforms, we find

$$\hat{\mu}_l(\xi) = (2\pi)^{-1/2}\left[\prod_{j=1}^{l} m_0(2^{-j}\xi)\right]\left[\frac{\sin(2^{-l-1}\xi)}{2^{-l-1}\xi}\right]^2.$$

From Lemma 3.1 it follows that $\hat{\mu}_l \to \hat{\eta}_\infty$, uniformly on compact sets. This implies that, for all $\delta > 0$, and for all $R > 0$, we can find l_0 such that, for all $l \geqq l_0$,

$$\int_{|\xi| \leqq R} d\xi \, |\hat{\mu}_l(\xi) - \hat{\eta}_\infty(\xi)| \leqq \delta.$$

On the other hand, $\hat{\eta}_\infty \in L^1$ since $B < 2^{N-1}$. It follows that for all $\delta > 0$ there exists R such that

$$\int_{|\xi| \geqq R} d\xi \, |\hat{\eta}_\infty(\xi)| \leqq \delta.$$

L^1-convergence of $\hat{\mu}_l$ to $\hat{\eta}_\infty$, which implies pointwise convergence of μ_l to η_∞, will then follow if we can prove that, for all $\delta > 0$, there exist R and l_0 large enough, so that, for all $l \geqq l_0$,

$$\int_{|\xi| \geqq R} d\xi \, |\hat{\mu}_l(\xi)| \leqq \delta.$$

3. We need thus to evaluate the integral

$$\int_{|\xi| \geqq R} d\xi \, |P_l(\xi)| \left| \frac{\sin(2^{-l-1}\xi)}{2^{-l-1}\xi} \right|^2,$$

where $P_l(\xi) = \prod_{j=1}^{l} m_0(2^{-j}\xi)$. To do this, we split the integral into two parts, namely $|\xi| \geqq 2^l\pi$ and $R \leqq |\xi| \leqq 2^l\pi$. To evaluate these two parts, we shall use the following three properties of P_l:

(i) $|P_l(\xi)| \leqq 1$, $\qquad$ (since $|m_0(\xi)| \leqq 1$),

(ii)
$$|P_l(\xi)| \leqq \left[\prod_{j=1}^{l} |\cos(2^{-j}\xi)| \right]^N \prod_{j=1}^{l} |\mathscr{F}(2^{-j}\xi)|$$
$$\leqq C \left| \frac{2^{-l}\sin\frac{1}{2}\xi}{\sin(2^{-l-1}\xi)} \right|^N (1 + |\xi|)^\beta,$$

where $\beta = \log B/\log 2$ (use the proof of Lemma 3.2) and

(iii) P_l is periodic, with period $2^{l+1}\pi$.

4. We

Choose λ

(3.40)

Now

where C_1

On the

Putting

implies th

(3.41)

4. We concentrate first on $|\xi| \geq 2^l\pi$. Using the periodicity of P_l, we find

$$\int_{|\xi| \geq 2^l\pi} d\xi\, |P_l(\xi)| \left| \frac{\sin(2^{-l-1}\xi)}{2^{-l-1}\xi} \right|^2$$

$$= \sum_{k \neq 0} \int_{|\xi| \leq 2^l\pi} d\xi\, |P_l(\xi)| \frac{|\sin(2^{-l-1}\xi)|^2}{|2^{-l-1}\xi + k\pi|^2}$$

$$\leq C \int_{|\xi| \leq 2^l\pi} d\xi\, |P_l(\xi)|\, |\sin(2^{-l-1}\xi)|^2.$$

Choose $\lambda = 2^{-\alpha l}$, with $\alpha \in]0,1[$ to be fixed later. Then

$$\int_{|\xi| \leq 2^l\pi} d\xi\, |P_l(\xi)|\, |\sin(2^{-l-1}\xi)|^2 \tag{3.40}$$

$$\leq \tfrac{1}{4}\lambda^2 \int_{|\xi| \leq 2^l\lambda} d\xi\, |P_l(\xi)| + \int_{2^l\lambda \leq |\xi| \leq 2^l\pi} d\xi\, |P_l(\xi)|.$$

Now

$$\int_{|\xi| \leq 2^l\lambda} d\xi\, |P_l(\xi)|$$

$$\leq \int_{|\xi| \leq 1} d\xi\, |P_l(\xi)| + C \int_{1 \leq |\xi| \leq 2^l\lambda} d\xi\, (1 + |\xi|)^\beta |2^l \sin(2^{-l-1}\xi)|^{-N}$$

$$\leq 1 + 2^N C \int_1^\infty dx\, (1+x)^\beta x^{-N} = C_1,$$

where C_1 is finite because $N - \beta > 1$.

On the other hand,

$$\int_{2^l\lambda \leq |\xi| \leq 2^l\pi} d\xi\, |P_l(\xi)|$$

$$\leq C 2^{-lN}(1 + 2^l\pi)^\beta 2^l \int_\lambda^\pi dx\, |\sin \tfrac{1}{2}x|^{-N}$$

$$\leq C_2 2^{l(1+\beta-N)}\lambda^{-N}.$$

Putting it all together, and choosing $\alpha = (N - \beta - 1)/(N+2) \in]0,1[$, this implies that (3.40) is

$$\leq C_3 2^{-2l(N-\beta-1)/(N+2)}. \tag{3.41}$$

This clearly tends to zero for $l \to \infty$.

5. We now evaluate the integral of $|\hat{\mu}_l|$ over $R \leq |\xi| \leq 2^l\pi$. Since $|\sin x| \geq 2|x|/\pi$ for $|x| \leq \frac{1}{2}\pi$, we find

$$\int_{R \leq |\xi| \leq 2^l\pi} d\xi\, |P_l(\xi)| \left|\frac{\sin(2^{-l-1}\xi)}{2^{-l-1}\xi}\right|^2$$

$$\leq 4C \int_{R \leq |\xi| \leq 2^l\pi} d\xi\, (1+|\xi|)^{\beta} |\xi|^{-2} |\sin\tfrac{1}{2}\xi|^{N} \left|\frac{2^{-l}}{\sin(2^{-l-1}\xi)}\right|^{N-2}$$

$$\leq 4C\pi^{N-2} \int_R^{\infty} dx\, (1+x)^{\beta} x^{-N}.$$

Since $N - \beta - 1 > 0$, this tends to zero for $R \to \infty$, uniformly in l. Together with (3.41) this proves that

$$\int_{|\xi| \geq R} d\xi\, |\hat{\mu}_l(\xi)|$$

can be made as small as wanted, by choosing l and R large enough. As pointed out in point 2, this proves $\|\hat{\mu}_l - \hat{\eta}_\infty\|_{L^1} \to_{l \to \infty} 0$.

6. We have thus proved that $\mu_l \to \eta_\infty$, pointwise. In fact, we can even show a little bit more. The same arguments (points 2 $\to$ 5) as above can be stretched a little to prove

$$\int d\xi\, (1+|\xi|)^{\lambda} |\hat{\eta}_\infty(\xi)| < \infty$$

and

$$\int d\xi\, (1+|\xi|)^{\lambda} |\hat{\eta}_\infty(\xi) - \hat{\mu}_l(\xi)| \underset{l \to \infty}{\longrightarrow} 0,$$

where

$$\lambda = \tfrac{1}{2}(N - \beta - 1) > 0.$$

Consequently, η_∞ is λ-Lipschitz,

$$|\eta_\infty(x) - \eta_\infty(y)| \leq C|x-y|^{\lambda},$$

and the convergence $\mu_l \to \eta_\infty$ is uniform on compact sets.

7. Finally, we only need to show that pointwise convergence of the μ_l implies pointwise convergence of the η_l. The two functions μ_0 and η_0 agree on integers,

$$\mu_0(0) = \eta_0(0) = 1,$$

$$\mu_0(k) = \eta_0(k) = 0 \quad \text{for} \quad k \in \mathbf{Z},\ k \neq 0.$$

Using the recursion relation (3.39), which both the μ_l and the η_l satisfy, one sees that this implies, for all $l \in \mathbf{N}$,

$$\eta_l(2^{-l}k) = \mu_l(2^{-l}k) \quad \text{for all} \quad k \in \mathbf{Z}.$$

Let $x \in \mathbf{R}$ be arbitrary. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| \leqq \delta \Rightarrow |\eta_\infty(x) - \eta_\infty(y)| \leqq \tfrac{1}{2}\varepsilon.$$

There also exists l_0 such that, for all $l \geqq l_0$, and all $y \in [x - \delta, x + \delta]$, one has

$$|\eta_\infty(y) - \mu_l(y)| \leqq \tfrac{1}{2}\varepsilon.$$

Choose $l \geqq l_1 = \max(l_0, -\ln\delta/\ln 2)$. Since η_l is piecewise constant, with step width 2^{-l}, it follows that there exists $k \in \mathbf{Z}$ such that

$$|x - 2^{-l}k| \leqq 2^{-l} \leqq \delta$$

and

$$\eta_l(x) = \eta_l(2^{-l}k) = \mu_l(2^{-l}k).$$

Hence

$$|\eta_l(x) - \eta_\infty(x)| \leqq |\mu_l(2^{-l}k) - \eta_\infty(2^{-l}k)| + |\eta_\infty(2^{-l}k) - \eta_\infty(x)| \leqq \varepsilon.$$

Since ε was arbitrary, this shows that η_l converges pointwise to η_∞, for $l \to \infty$.

Remarks. 1. Using only slightly modified arguments, one proves, under the same conditions (in fact, only $B < 2^{N-1/2}$ is needed) that $\eta_l \to \eta_\infty$ in L^2, for $l \to \infty$. One simply replaces the L^1-estimates for $\eta_\infty - \mu_l$ by L^2-estimates for $\eta_\infty - \eta_l$ (no intermediary μ_l are needed).

2. As noted above, it is sufficient that

$$\sum_n |h(n)|\,|n|^{N+\varepsilon} < \infty$$

to ensure (3.37).

3. The $h(n)$ of Example 3.1 do not satisfy the conditions of the proposition, since in this case

$$m_0(\xi) = \tfrac{1}{2}(1 + e^{i\xi}),$$

hence $N = 1$, $B = |\mathscr{F}(\xi)| = 1$, and therefore $B = 2^{N-1}$. However, in this case one checks directly that

$$\eta_l = \chi_{[-2^{-l-1}, 1-2^{-l-1}[}.$$

The limit η_∞ is not continuous in this case, $\eta_\infty = \chi_{[0,1[}$, but the pointwise convergence $\eta_l \to \eta_\infty$ still holds a.e.

4. The coefficients $h(n)$ defined by

$$h(0) = h(3) = 2^{-1/2},$$
$$h(n) = 0 \quad \text{otherwise},$$

satisfy all the "discrete" conditions of subsection 3A, but do not satisfy the conditions in the last proposition (for the same reason as the $h(n)$ of Example 3.1). In this case, however, the pointwise convergence of the η_l fails on a whole interval. It is easy to check that, for any l, the η_l take only two values, 0 and 1. (The easiest way to check this is to use the "graphical" construction (2.40) of the η_l—see subsection 2B and Figure 3.) On the other hand,

$$m_0(\xi) = \tfrac{1}{2}(1 + e^{3i\xi}),$$

hence

$$\hat{\eta}_\infty(\xi) = (2\pi)^{-1/2}\prod_{j=1}^{\infty} m_0(2^{-j}\xi) = (2\pi)^{-1/2} e^{3i\xi/2}\,\frac{\sin\frac{3}{2}\xi}{\frac{3}{2}\xi}$$

or

$$\eta_\infty = \tfrac{1}{3}\chi_{[0,3]}.$$

There is therefore no pointwise convergence for any x between 0 and 3. The L^2-convergence fails too, since $\|\eta_\infty\|_{L^2}^2 = \frac{1}{3}$, whereas for all finite l, η_l is the characteristic function of a union of intervals, and hence $\|\eta_l\|_{L^2}^2 = \|\eta_l\|_{L^1} = \hat{\eta}_l(0) = 1$.

5. Only two values of ν, in Example 3.2, lead to coefficients $h(n)$ that satisfy the conditions of the proposition. They correspond to $m_0(\xi)$ divisible by $(1 + e^{i\xi})^2$. As noted above, all $m_0(\xi)$ satisfying the discrete conditions in subsection 3A are divisible by $(1 + e^{i\xi})$ (see Remark 5 at the end of subsection 3A). In Example 3.2, extra divisibility by another factor $(1 + e^{i\xi})$ leads to the condition

$$h(1) - h(3) = 2h(0),$$

or

$$\nu = \pm 1/\sqrt{3}.$$

The corresponding $h(0), \cdots, h(3)$ are

$$\begin{aligned} h(0) &= (1 \mp \sqrt{3})/(4\sqrt{2}), \\ h(1) &= (3 \mp \sqrt{3})/(4\sqrt{2}), \\ h(2) &= (3 \pm \sqrt{3})/(4\sqrt{2}), \\ h(3) &= (1 \pm \sqrt{3})/(4\sqrt{2}). \end{aligned} \tag{3.42}$$

We shall come back to these $h(n)$ later.

With Proposition 3.3 we have completed our program of writing a set of explicit conditions on the $h(n)$, $g(n)$, without reference to a multiresolution analysis background, which make Mallat's algorithm work, and which moreover lead to filters with sufficient "regularity".

In the case where the $h(n)$, $g(n)$ are calculated starting from a multiresolution analysis (see subsection 2C), one has

$$h(n) = \langle \phi_{10}, \phi_{0n} \rangle,$$

or

$$\phi_{10} = \sum_n h(n) \phi_{0n},$$

i.e.,

$$\phi(\tfrac{1}{2}x) = 2^{1/2} \sum_n h(n) \phi(x - n).$$

This is equivalent to

$$\hat{\phi}(\xi) = 2^{-1/2} \sum_n h(n) e^{in\xi/2} \hat{\phi}(\tfrac{1}{2}\xi) = m_0(\tfrac{1}{2}\xi) \hat{\phi}(\tfrac{1}{2}\xi).$$

It follows that

$$\hat{\phi}(\xi) = \left[\prod_{j=1}^{\infty} m_0(2^{-j}\xi) \right] \hat{\phi}(0), \tag{3.43}$$

or, since $\hat{\phi}(0) = (2\pi)^{-1/2} \int dx\, \phi(x) = (2\pi)^{-1/2}$ (see (2.18)),

$$\phi(x) = \eta_\infty(x). \tag{3.44}$$

As pointed out in subsection 2B, the $\eta_l = T^l \chi_{[-1/2,1/2[}$ can also be computed via a different recursion, (2.40), which we shall call the "graphical" recursion, and which lies at the basis of the graphical construction technique illustrated by Figure 3. It follows from (3.44) that, in the case where the $h(n)$ are derived from a multiresolution analysis framework, the graphical construction by iteration (see Figure 3, where the $h(n)$ now play the role of the $w(n)$) is therefore nothing but a reconstruction of the function ϕ; in the limit for $l \to \infty$, finer and finer detail is achieved for increasing l.

3.C. Equivalence between the discrete conditions and multiresolution analysis. So far we have formulated conditions, directly on the $h(n)$, which ensure that S. Mallat's algorithm works (with these coefficients), and has regularity (in the sense given to it at the end of subsection 2B, or in subsection 3B). We have seen for every condition how the coefficients $h(n)$ computed from a multiresolution analysis fit into the picture. The main result of this subsection is that these multiresolution-based examples are the *only* ones. It turns out that *any* sequence of $h(n)$ satisfying the conditions in subsections 3A and 3B corresponds to a multiresolution analysis. The function η_∞ defined by (3.26) is then exactly the function ϕ from the multiresolution structure.

To prove this equivalence, we start from a sequence $h(n)$ satisfying (3.18), (3.19) and (3.27). We also assume that the function $m_0(\xi) = 2^{-1/2}\Sigma_n h(n)e^{in\xi}$ satisfies all the conditions in Proposition 3.3. We then define, as in (3.17),

$$g(n) = (-1)^n h(-n+1), \tag{3.45}$$

and, as in (3.44),

$$\phi(x) = \eta_\infty(x),$$

or

$$\hat{\phi}(\xi) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\xi). \tag{3.46}$$

From the proof of Proposition 3.3 we know that ϕ is a bounded, uniformly continuous function; since $\hat{\phi} \in L^1 \cap L^\infty$, one also has $\phi \in L^2$. We define, in accordance with (2.16),

$$\psi(x) = \sqrt{2} \sum_n g(n)\phi(2x - n). \tag{3.47}$$

Since $\Sigma_n |g(n)| = \Sigma_n |h(n)| < \infty$, it follows that

$$|\hat{\psi}(\xi)| \leq 2^{-1/2} \sum_n |h(n)| \cdot |\hat{\phi}(\tfrac{1}{2}\xi)|.$$

All the estimates of subsection 3B on η_∞ carry over, therefore, to ψ, and one finds that ψ is a bounded, uniformly continuous L^2-function. As before, we define $\psi_{jk}(x) = 2^{-j/2}\psi(2^{-j}x - k)$, and $\phi_{jk}(x) = 2^{-j/2}\phi(2^{-j}x - k)$. The definitions (3.46) and (3.47) immediately imply

$$\phi_{jk} = \sum_n h(n - 2k)\phi_{j-1\,n}, \tag{3.48}$$

$$\psi_{jk} = \sum_n g(n - 2k)\phi_{j-1\,n}. \tag{3.49}$$

We shall prove that the ψ_{jk} constitute an orthonormal basis of $L^2(\mathbb{R})$. In a first step we prove some orthogonality relations.

LEMMA 3.4. *Let* $h(n)$ *satisfy* (3.18), (3.19), (3.28) *and the conditions in Proposition* 3.3. *Let* $g(n)$, ϕ, ψ *be defined by* (3.45), (3.46), (3.47), *respectively.*

Then ϕ, $\psi \in L^2(\mathbf{R})$, *and, for all* j, k, $k' \in \mathbf{Z}$,

$$\langle \psi_{jk}, \psi_{jk'} \rangle = \delta_{kk'}, \tag{3.50}$$

$$\langle \psi_{jk}, \phi_{jk'} \rangle = 0, \tag{3.51}$$

$$\langle \phi_{jk}, \phi_{jk'} \rangle = \delta_{kk'}. \tag{3.52}$$

Remark. Note that (3.50)–(3.52) are restricted to one j-level at a time. The orthogonality between j-levels will follow from Lemma 3.5.

Proof: 1. Let η_l be defined as in Proposition 3.3,

$$\eta_l = T^l \chi_{[-1/2,1/2[},$$

with

$$(Tf)(x) = \sqrt{2} \sum_n h(n) f(2x - n). \tag{3.53}$$

For reasons which will become obvious, we add an index 0 to η_l,

$$\eta_{l,0} = \eta_l.$$

For arbitrary $k \in \mathbf{Z}$, we define

$$\eta_{l,k} = (T_k)^l \chi_{[-1/2+k,1/2+k[}$$

with $(T_k f)(x) = \sqrt{2} \sum_n h(n) f(2x - n - k)$. Due to the translations over k, built into $\eta_{0,k}$ as well as into T_k, $\eta_{l,k}$ is just a translated version of $\eta_{l,0}$. This can easily be checcked by induction,

$$\eta_{0,k}(x) = \chi_{[-1/2+k,1/2+k[}(x) = \eta_{0,0}(x - k)$$

and

$$\eta_{l,k}(x) = \sqrt{2} \sum_n h(n) \eta_{l-1,k}(2x - n - k)$$

$$= \sqrt{2} \sum_n h(m) \eta_{l-1,0}(2x - 2k - n)$$

$$= \eta_{l,0}(x - k).$$

Since (see Remark 1 following Proposition 3.3) $\|\eta_{l,0} - \phi\|_{L^2} \to 0$ for $l \to \infty$, it follows that $\|\eta_{l,k} - \phi_{0k}\|_{L^2} \to 0$ for $l \to \infty$.

2. Since $\hat{\eta}_{l,0}(\xi) = \left[\prod_{j=1}^{l} m_0(2^{-j}\xi)\right] \hat{\eta}_{0,0}(2^{-l}\xi)$, and since $|m_0(\xi)| \leq 1$ and $\eta_{0,0} \in L^2$, it follows that all the $\eta_{l,k}$ are in L^2.

3. For fixed l, the different $\eta_{l,k}$ are orthonormal. This can again be proved by induction. By translation invariance, it is sufficient to prove that $\langle \eta_{l,k}, \eta_{l,k'} \rangle = \delta_{kk'}$ for $k' = 0$. We have

$$\langle \eta_{0,k}, \eta_{0,0} \rangle = \int_{-1/2}^{1/2} dx\, \chi_{[-1/2+k,1/2+k[}(x) = \delta_{k0}$$

and

$$\begin{aligned} \langle \eta_{l,k}, \eta_{l,0} \rangle &= 2 \sum_{n,m} h(n) h(m) \int dx\, \eta_{l-1,k}(2x - n - k) \eta_{l-1,0}(2x - m) \\ &= 2 \sum_{n,m} h(n) h(m) \int dx\, \eta_{l-1,2k+n-m}(2x) \eta_{l-1}(2x) \\ &= \sum_{n,m} h(n) h(m)\, \delta_{0,2k+n-m} = \sum_n h(n) h(m + 2k) \\ &= \delta_{k,0} \qquad \text{(by (3.18))}. \end{aligned}$$

By induction it follows that $\langle \eta_{l,k}, \eta_{l,k'} \rangle = \delta_{kk'}$ for all l, k, k'.

4. It follows immediately that

$$\begin{aligned} \langle \phi_{jk}, \phi_{jk'} \rangle &= 2^{-j} \int dx\, \phi(2^{-j}x - k) \phi(2^{-j}x - k') \\ &= \int dx\, \phi(x) \phi(x - k' + k) \\ &= \lim_{l \to \infty} \langle \eta_{l,0}, \eta_{l,k'-k} \rangle = \delta_{kk'}. \end{aligned}$$

5. With $g(n)$ defined by (3.45), the conditions (3.18), (3.19) on the $h(n)$ imply (see subsection 3A)

$$\sum_n g(n - 2k) h(n - 2l) = 0, \tag{3.54}$$

$$\sum_n g(n - 2k) g(n - 2l) = \delta_{kl}. \tag{3.55}$$

Hence, by (3.48) and (3.49),

$$\begin{aligned} \langle \psi_{jk}, \phi_{jk'} \rangle &= \sum_{n,n'} g(n - 2k) h(n' - 2k') \langle \phi_{j-1n}, \phi_{j-1n'} \rangle \\ &= \sum_n g(n - 2k) h(n - 2k') = 0, \end{aligned}$$

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$$\langle \psi_{jk}, \psi_{jk'} \rangle = \sum_{m,n'} g(m-2k) g(n'-2k') \langle \phi_{j-1n}, \phi_{j-1n'} \rangle$$

$$= \sum_{n} g(n-2k) g(n-2k') = \delta_{kk'}.$$

The "discrete orthogonality condition" (3.18) plays a crucial role in this proof. In the terminology of subsection 3A, (3.18) is equivalent to $HH^* = \mathbb{1}$, where H^* is the bounded l^2-operator (see subsection 3A)

$$(H^* a)^n = \sum_k h(n-2k) a_k.$$

This implies that H^*, as an operator from l^2 to l^2, preserves orthogonality of sequences. The operator T_H defined by (3.24) was in fact constructed to exactly reproduce, when acting on $\chi_{[-1/2,1/2]}$ and its iterates, the action of H^* on the sequence e ($e_n = \delta_{n0}$) and its iterates (see subsection 2B). This implies that repeated application of T_H preserves the orthogonality of the $\eta_{0,k}$. This is what makes the above proof work.

In the following lemma we prove that the ψ_{jk} constitute a tight frame (see Section 1, or (3.57) below). Again, the crucial ingredient will be one of the discrete identities which follow from the conditions on $h(n)$, $g(n)$. From subsection 3A we know that, with $g(n)$ as defined by (3.45), and with $h(n)$ satisfying all the conditions above,

$$\sum_k [h(n-2k)h(m-2k) + g(n-2k)g(m-2k)] = \delta_{mn}$$

(this can also be derived directly from (3.18) and (3.45)). It follows that (use (3.48), (3.49))

$$\sum_k [h(m-2k)\phi_{jk} + g(m-2k)\psi_{jk}] = \phi_{j-1m}. \tag{3.56}$$

This, of course, already points towards multiresolution analysis (see subsection 2A).

LEMMA 3.5. *Let $h(n)$, $g(n)$, ϕ, ψ be as in Lemma 3.4. Then, for all $f \in L^2(\mathbf{R})$,*

$$\sum_{j,k \in \mathbf{Z}} |\langle \psi_{jk}, f \rangle|^2 = \|f\|^2. \tag{3.57}$$

Proof: 1. Take any $f \in C_0^\infty$. Then, since $\phi \in L^2$, $\Sigma_n |\langle \phi_{jn}, f \rangle|^2$ converges, for any $j \in \mathbf{Z}$. Moreover, by (3.56),

$$\sum_n |\langle \phi_{j-1\,n}, f\rangle|^2 = \sum_{n,k,l} \Big[h(n-2k)h(n-2l)\langle \phi_{jk}, f\rangle\langle f, \phi_{jl}\rangle + 2h(n-2k)g(n-2l)\,\mathcal{R}e\big(\langle \phi_{jk}, f\rangle\langle f, \psi_{jl}\rangle\big) + g(n-2k)g(n-2l)\langle \psi_{jk}, f\rangle\langle f, \psi_{jl}\rangle \Big]$$

$$= \sum_k \Big[|\langle \phi_{jk}, f\rangle|^2 + |\langle \psi_{jk}, f\rangle|^2 \Big],$$

where we have used (3.18), (3.54) and (3.55).

2. By iteration, one has, for all $N \in \mathbf{N}$,

$$\sum_n |\langle \phi_{-Nn}, f\rangle|^2 = \sum_k |\langle \phi_{Nk}, f\rangle|^2 + \sum_{j=-N}^{N} \sum_k |\langle \psi_{jk}, f\rangle|^2. \tag{3.58}$$

In this expression we shall let N tend to ∞.

3. We first concentrate on $\Sigma_k |\langle \phi_{Nk}, f\rangle|^2$. Let us suppose, for the sake of definiteness, that $\operatorname{supp} f \subset [-2^{n_0}, 2^{n_0}]$. Take $N \geqq n_0 + 1$, so that the translation steps in the $\phi_{Nk}(x) = \phi_{N0}(x - 2^N k)$ are larger than $|\operatorname{supp} f|$. On the other hand, for any $\varepsilon > 0$ there exists $k_0 \in \mathbf{N}$ such that

$$\int_{|x| \geq k_0} dx\, |\phi(x)|^2 \leqq \varepsilon.$$

Then

$$\sum_k |\langle \phi_{Nk}, f\rangle|^2$$

$$= \sum_{|k| \leq k_0} |\langle \phi_{Nk}, f\rangle|^2 + \sum_{|k| \geq k_0+1} |\langle \phi_{Nk}, f\rangle|^2$$

$$\leqq (2k_0+1)2^{-N}\|\phi\|_\infty^2 \|f\|_1^2 + \|f\|_2^2 2^{-N} \sum_{|k| \geq k_0+1} \int_{|x| \leq 2^{n_0}} dx\, |\phi(2^{-N}x - k)|^2$$

$$\leqq 2^{-N}(2k_0+1)\|\phi\|_\infty^2 \|f\|_1^2 + \varepsilon \|f\|_2^2.$$

By choosing ε and N appropriately, this can be made arbitrarily small. Hence

$$\sum_k |\langle \phi_{Nk}, f\rangle|^2 \underset{N\to\infty}{\longrightarrow} 0. \tag{3.59}$$

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4. We now concentrate on $\sum_k |\langle \phi_{-Nk}, f\rangle|^2$. By means of the Poisson formula this can be rewritten as

$$\sum_k |\langle \phi_{-Nk}, f\rangle|^2$$

$$= 2\pi \sum_{l\in\mathbb{Z}} \int d\xi\, \hat\phi(2^{-N}\xi)\overline{\hat\phi(2^{-N}\xi + 2\pi l)\,\hat f(\xi)}\,\hat f(\xi + 2\pi l 2^N)$$

$$= 2\pi \int d\xi |\hat\phi(2^{-N}\xi)|^2 |\hat f(\xi)|^2 + R. \tag{3.60}$$

Here

$$|R| \leq \sum_{l\neq 0} \int d\xi\, |\hat f(\xi)|\,|\hat f(\xi + 2\pi l 2^N)|,$$

because $|\hat\phi(\xi)| = (2\pi)^{-1/2}\prod_{j=1}^{\infty}|m_0(2^{-j}\xi)| \leq (2\pi)^{-1/2}$, since $|m_0(\xi)| \leq 1$. Since $f \in C_0^\infty$, we can find C such that

$$|\hat f(\xi)| \leq C(1 + |\xi|)^{-3}.$$

An easy estimation then leads to

$$|R| \leq C' 2^{-3N/2}.$$

This tends to zero for $N \to \infty$.

5. We now examine the first term in (3.60). One has

$$|\hat\phi(\xi) - \hat\phi(0)| = (2\pi)^{-1/2}\left|\prod_{j=1}^{\infty} m_0(2^{-j}\xi) - \prod_{j=1}^{\infty} m_0(0)\right|$$

$$\leq (2\pi)^{-1/2} \sum_{j=1}^{\infty} |m_0(2^{-j}\xi) - m_0(0)|,$$

since $|m_0(\zeta)| \leq 1$ for all $\zeta \in \mathbb{R}$. But

$$|m_0(\zeta) - m_0(0)| \leq 2^{-1/2} \sum_n |h(n)|\,|e^{in\zeta} - 1|$$

$$\leq C|\zeta|^{\epsilon},$$

where we have used (3.19) and $|e^{i\alpha} - 1| \leq C_\epsilon |\alpha|^\epsilon$ (we assume $0 < \epsilon \leq 1$). Hence,

$$|\hat\phi(\xi) - \hat\phi(0)| \leq (2\pi)^{-1/2} C \sum_{j=1}^{\infty} |2^{-j}\xi|^\epsilon \leq C'|\xi|^\epsilon.$$

Consequently, using $\hat{\phi}(0) = (2\pi)^{-1/2}$, we find

$$2\pi \int d\xi \, |\hat{\phi}(2^{-N}\xi)|^2 |\hat{f}(\xi)|^2$$

$$\leq \int d\xi \, |\hat{f}(\xi)|^2 + 2\pi \int d\xi \left[2C'|2^{-N}\xi|^{\varepsilon} + C'^2|2^{-N}\xi|^{2\varepsilon}\right] |\hat{f}(\xi)|^2$$

$$= \|f\|^2 + C'' 2^{-N\varepsilon} \int d\xi (1 + |\xi|^{2\varepsilon}) |\hat{f}(\xi)|^2.$$

This converges to $\|f\|^2$ as $N \to \infty$. Hence

$$\sum_k |\langle \phi_{-Nk}, f\rangle|^2 \xrightarrow[N\to\infty]{} \|f\|^2. \tag{3.61}$$

6. Putting together (3.58), (3.60) and (3.61) shows that, for all $f \in C_0^\infty(\mathbf{R})$,

$$\sum_{j,k} |\langle \psi_{jk}, f\rangle|^2 = \|f\|^2_{L^2}. \tag{3.62}$$

Since $C_0^\infty(\mathbf{R})$ is dense in $L^2(\mathbf{R})$, (3.62) extends to all $f \in L^2(\mathbf{R})$.

Since $\|\psi\| = 1$ (this is a special case of (3.50), with $j = k = k' = 0$), (3.57) implies that the ψ_{jk} constitute an orthonormal basis. This completes the proof of the main theorem of this section.

THEOREM 3.6. *Let $h(n)$ be a sequence such that*

(i) $\Sigma_n |h(n)|\,|n|^{\varepsilon} < \infty$ *for some* $\varepsilon > 0$,
(ii) $\Sigma_n h(n - 2k) h(n - 2l) = \delta_{kl}$,
(iii) $\Sigma h(n) = 2^{1/2}$.

Suppose also that $m_0(\xi) = 2^{-1/2}\Sigma_n h(n) e^{in\xi}$ *can be written as*

$$m_0(\xi) = \left[\tfrac{1}{2}(1 + e^{i\xi})\right]^N \left[\sum_n f(n) e^{in\xi}\right],$$

where

(iv) $\Sigma_n |f(n)|\,|n|^{\varepsilon} < \infty$ *for some* $\varepsilon > 0$,
(v) $\sup_{\xi \in \mathbf{R}} |\Sigma_n f(n) e^{in\xi}| < 2^{N-1}$.

Define

$$g(n) = (-1)^n h(-n + 1),$$

$$\hat{\phi}(\xi) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\xi),$$

$$\psi(x) = 2^{1/2} \sum_n g(n) \phi(2x - n).$$

Then the $\phi_{jk}(x) = 2^{-j/2}\phi(2^{-j}x - k)$ *define a multiresolution analysis (in the sense of subsection* 2A); *the* ψ_{jk} *are the associated orthonormal wavelet basis.*

Remarks. 1. As we already said in the introduction, this theorem is also proved in [19], under slightly different conditions. The growth restrictions (3.37) and (3.38) on the $h(n)$ are replaced, in [19], by the condition that $\inf_{|\xi| \le \pi/2} |m_0(\xi)| > 0$. Together with $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1$, $m_0(0) = 1$, this condition implies that the ϕ_{jk}, with ϕ defined as above, define a multiresolution analysis. The function ϕ may, however, still be very irregular; the coefficients $h(n)$ used in Figure 4, e.g., satisfy the positivity condition of [19], but are clearly not associated with a regular ϕ. In the present paper, we emphasized regularity of the discrete filters; once regularity is ensured by means of conditions (3.36)–(3.38), equivalence with regular multiresolution analysis follows. Consequently, the techniques of our proofs and the proofs in [19] are quite different. The basic intuition for the present proof was mainly graphical. As explained above, the orthogonality of the ϕ_{0k} follows naturally, given our "graphical" construction, from the discrete conditions. Similarly, (3.60) can be understood graphically.

2. At the end of subsection 3B (Remark 3) we mentioned the link between the present construction and the "conjugated quadrature filters" of Smith and Barnwell [24]. Any of their conjugated quadrature filters will satisfy all the conditions in subsection 3A. Provided they also satisfy the regularity condition in subsection 3B, they can be used to construct orthonormal wavelet bases. Since the goals of [24] are completely different however, most of the examples in [24] do not satisfy our regularity condition.

4. Orthonormal Bases of Wavelets with Compact Support

In subsection 2A we reviewed how orthonormal bases of wavelets can be constructed, starting from a multiresolution analysis framework. The basic ingredient there was a function ϕ such that (2.15) held, for some c_n, without even requiring the ϕ_{0n} to be orthogonal. Theorem 3.6 gives another recipe for constructing an orthonormal basis of wavelets (and the associated multiresolution analysis), this time from a sequence $(h(n))_{n \in \mathbf{Z}}$.

If this sequence has finite length, $h(n) = 0$ for $n < N_-$, or $n > N_+$, then the corresponding basic wavelet has compact support. This can be checked very easily from the graphical construction of ϕ (see Figures 2, 4), or from the recursive definition of the η_l,

$$\phi(x) = \lim_{l \to \infty} \eta_l(x), \tag{4.1}$$

$$\eta_l(x) = \sqrt{2} \sum_n h(n) \eta_{l-1}(2x - n), \tag{4.2}$$

$$\eta_0 = \chi_{[-1/2, 1/2[}. \tag{4.3}$$

The recursive definition of the η_l implies that all the η_l have compact support, $\text{supp}\, \eta_l \subset [N_{l,-}, N_{l,+}]$, with $N_{l,-} = \frac{1}{2}(N_{l-1,-} + N_-)$, and $N_{l,+} = \frac{1}{2}(N_{l-1,+} + N_+)$, while $N_{0,-} = -\frac{1}{2}$, $N_{0,+} = \frac{1}{2}$. Hence $H_{l,-} \to N_-$, $N_{l,+} \to N_+$ for $l \to \infty$, which implies that ϕ has compact support $\subset [N_-, N_+]$. Since only finitely many $g(n)$ are non-zero ($g(n) = 0$ for $n < -N_+ + 1$ or $n > -N_- + 1$), ψ also has compact support,

$$\text{supp}\, \psi \subset \left[\tfrac{1}{2}(1 - N_+ - N_-), \tfrac{1}{2}(1 + N_+ - N_-)\right].$$

In order to construct orthonormal bases of compactly supported wavelets, it suffices, therefore, to construct finite-length sequences $h(n)$ satisfying all the conditions of Theorem 3.6. An example of such a finite-length sequence is Example 3.2, with $\nu = \pm 1/\sqrt{3}$ (see Remark 5 following Proposition 3.3). In this case one finds (see (3.42)) $N_- = 0$, $N_+ = 3$, and

$$m_0(\xi) = \left[\tfrac{1}{2}(1 + e^{i\xi})\right]^2 \tfrac{1}{2}\left[(1 \mp \sqrt{3}) + (1 \pm \sqrt{3})e^{i\xi}\right]. \tag{4.4}$$

Since

$$\sup_{\xi \in \mathbb{R}} \tfrac{1}{2}\left|(1 \mp \sqrt{3}) + (1 \pm \sqrt{3})e^{i\xi}\right| = \sqrt{3} < 2,$$

the example (3.42) satisfies all the required conditions. The $h(n)$ given by (3.42) correspond, therefore, to an orthonormal basis of continuous wavelets. The basic wavelet has support width equal to $N_+ - N_- = 3$. Figure 5 shows the graphs of ϕ, ψ and their Fourier transforms, for this example. There are several striking features in Figure 5. First of all, it is obvious that even though ϕ and ψ are continuous, they are not very regular. There exist other constructions of compactly supported wavelet bases, in which ϕ and ψ have more regularity, at the cost of larger numbers of non-zero coefficients $h(n)$, which results in larger support widths for ψ, ϕ. For the family of examples we shall examine below, the support width of ψ, ϕ increases linearly with their regularity. Another striking feature of Figure 5 is the lack of any symmetry or antisymmetry axis for ψ, ϕ. This is quite unlike the Meyer wavelets (see [4]) or the Battle-Lemarié wavelets (see [16]). In all these (non-compactly supported) examples, ϕ is an even function, and ψ is symmetric around $x = \frac{1}{2}$. We shall see below that, except for the Haar basis (see (1.9) or Example 3.1), there exist *no* compactly supported wavelet bases in which ϕ is either symmetric or antisymmetric around any axis.

The plots of ψ and ϕ in Figure 5 (and later figures, for other examples) are made by direct implementation of the "graphical recursion algorithm" equivalent with (4.1)–(4.3) (see subsection 2B). This is much more efficient than Fourier transform of the infinite product (3.46) (see [26]). To plot Figure 5, only 8 iterations of type (2.40) were needed (i.e., η_8 is plotted rather than ϕ; the

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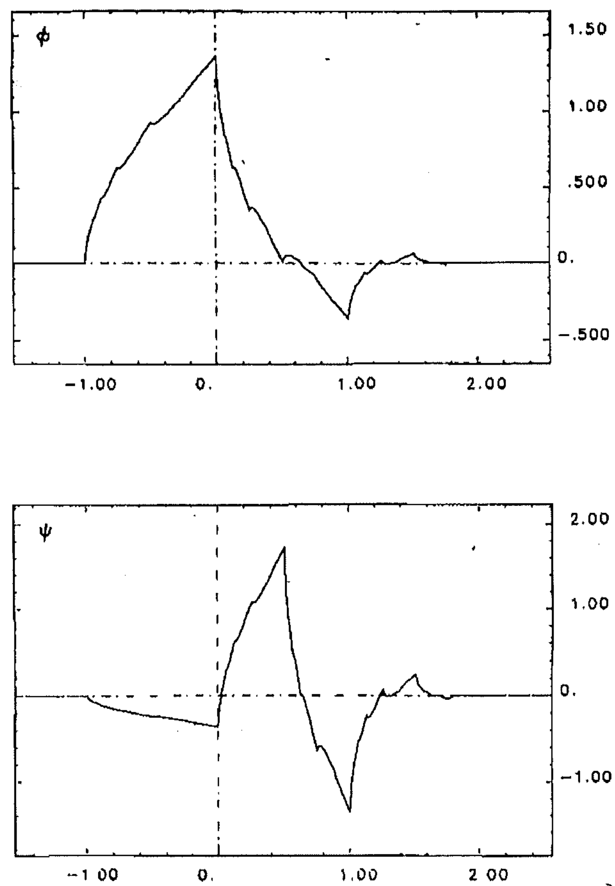
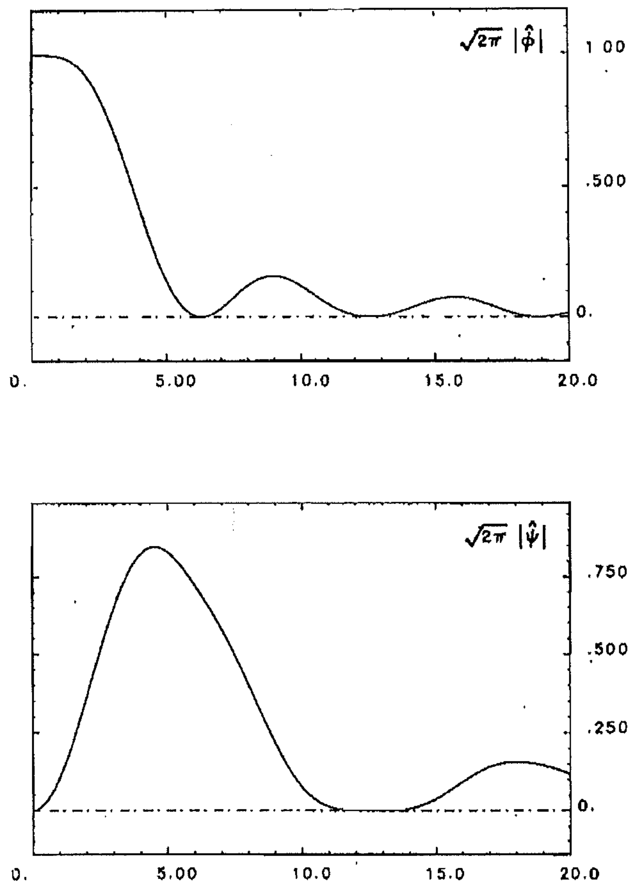
difference is not detectable at the scale of the figure). If more detail is wanted at any point (see Figure 6), it is possible to restrict to a neighborhood, and to locally iterate a few times more to obtain this detail.

In the following subsections we describe families of examples of compactly supported wavelet bases, and their properties. Henceforth, we shall always assume that only finitely many $h(n)$ are non-zero.



(a)

Figure 5. The functions ϕ, ψ, and the modulus of their Fourier transforms, $|\hat{\phi}|$, $|\hat{\psi}|$, for the orthonormal basis of compactly supported wavelets corresponding to the $h(n)$ in (3.42) (see text). Out of the two possibilities in (3.42) we choose the one corresponding to $\nu = -1/\sqrt{3}$ (i.e., $h(0) = (1 + \sqrt{3})/4\sqrt{2}$, etc.)



(b)

Figure 5. Continued

4.A. Lack of symmetry. Here we shall use again the notations $a(n), \cdots, d(n)$ (see (3.7)) and $\alpha(\xi), \cdots, \delta(\xi)$ (see (3.9)) introduced in subsection 3A. Let us define, for any trigonometric polynomial $P(\xi) = \Sigma_n p_n e^{in\xi}$, the two numbers

$$N_+(P) = \max\{n;\ p_n \neq 0\},$$

$$N_-(P) = \min\{n;\ p_n \neq 0\}.$$

One easily checks that

$$N_+(|P|^2) = -N_-(|P|^2) = N_+(P) - N_-(P).$$

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Since $|\alpha(\xi)|^2 + |\beta(\xi)|^2 = 1$ (see (3.14)), and $\alpha \neq 0$, $\beta \neq 0$ (see (3.15)), this implies

$$N_+(\alpha) - N_-(\alpha) = N_+(\beta) - N_-(\beta). \tag{4.5}$$

On the other hand, the definition (3.7) of the $a(n)$, $b(n)$, gives

$$N_+(m_0) = \max(2N_+(\alpha), 2N_+(\beta) + 1),$$

$$N_-(m_0) = \min(2N_-(\alpha), 2N_-(\beta) + 1).$$

Together with (4.5) this leads to

$$\begin{aligned} &N_+(m_0) - N_-(m_0) \\ &\quad = \max(2N_+(\alpha) - 2N_-(\beta) - 1, 2N_+(\beta) - 2N_-(\alpha) + 1). \end{aligned} \tag{4.6}$$

In any case, $N_+(m_0) - N_-(m_0)$ is an odd number.

If the function ϕ were symmetric around zero, $\phi(x) = \phi(-x)$, then $h(n) = h(-n)$ would follow. This would however imply $N_+(m_0) = -N_-(m_0)$, i.e., $N_+(m_0) - N_-(m_0) = 2N_+(m_0)$ would be even. Since this is in contradiction with (4.6), it follows that the function ϕ, associated with an orthonormal basis of wavelets with compact support, can never be an even function.

What about symmetry with respect to another point $\lambda \neq 0$? Suppose

$$\phi(\lambda + x) = \phi(\lambda - x),$$

where we can, without loss of generality, shift λ to the interval $]0, 1[$. Then it follows that

$$\hat{\phi}(\xi) = e^{2i\lambda\xi}\hat{\phi}(-\xi).$$

Because of the definition of $\hat{\phi}$ as the infinite product (3.46), this implies

$$m_0(\xi) = e^{2i\lambda\xi}m_0(-\xi).$$

Since both $m_0(\xi)$ and $m_0(-\xi)$ are trigonometric polynomials, this leaves only one possible value for λ, namely $\lambda = \frac{1}{2}$. Let us, therefore, assume that ϕ is symmetric with respect to $\frac{1}{2}$,

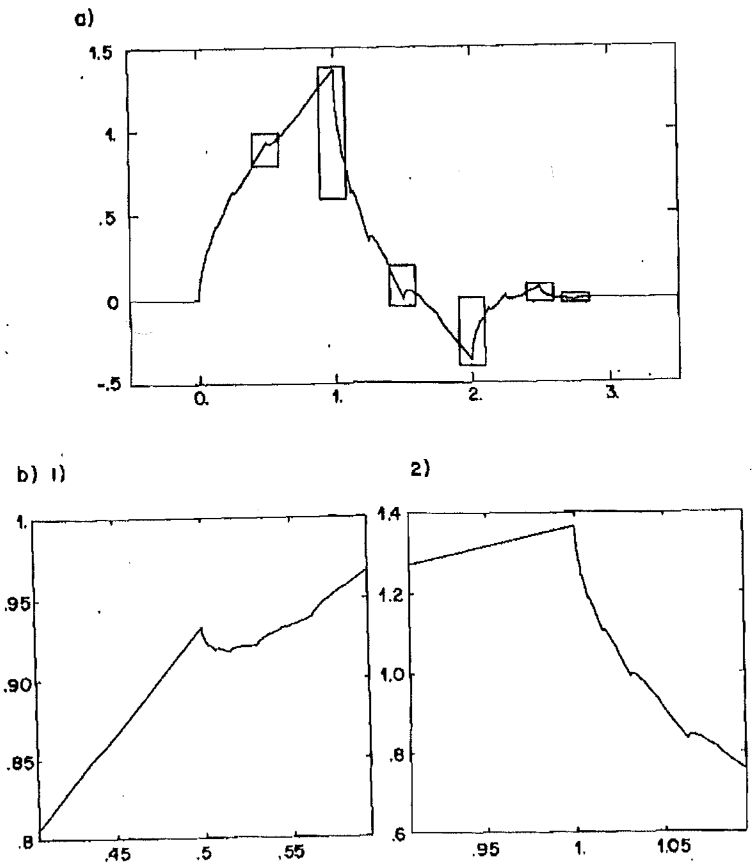
$$\phi(x + 1) = \phi(-x).$$

Then

$$h(2n + 1) = h(-2n),$$

or

$$b(n) = a(-n).$$



(a)

Figure 6. The function ϕ of Figure 5, and 6 local blow-ups
(a) The different zoom-in zones are shown on the graph of ϕ
(b) The blow-ups around 1) $x = .5$, 2) $x = 1$, 3) $x = 1.5$, 4) $x = 2.$, 5) $x = 2.5$, 6) $x + 2.75$.
The detail in these blow-ups illustrates the fractal, self-similar nature of this function ϕ.

Hence

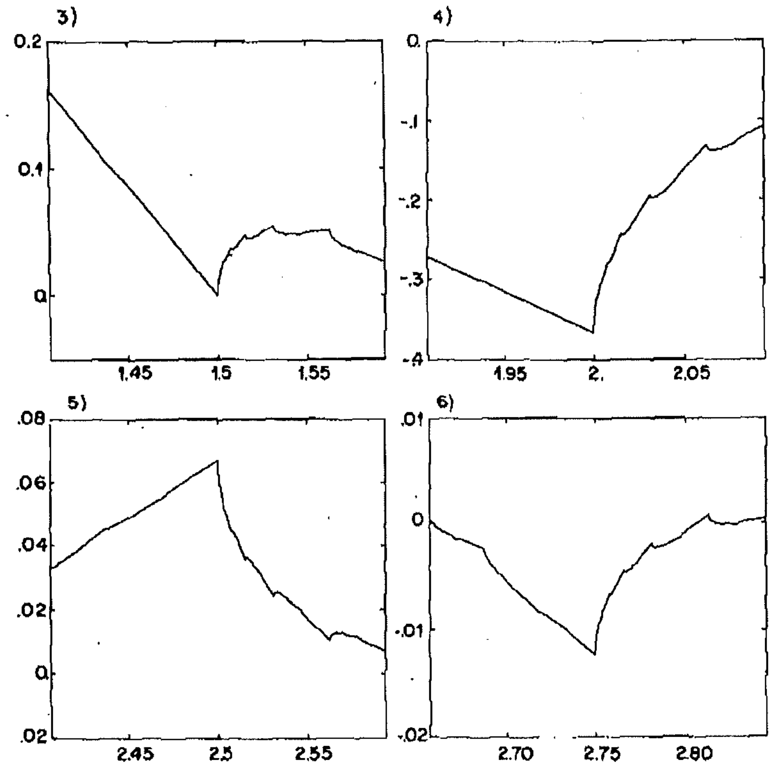
$$\beta(\xi) = \overline{\alpha(\xi)}\,.$$

Together with (3.14) this implies

$$2|\alpha(\xi)|^2 = 1,$$

or

$$a(n) = \pm 2^{-1/2}\delta_{nk} = b(-n) \quad \text{for some} \quad k \in \mathbb{N}.$$



(b)

Figure 6. Continued

We can, again without loss of generality, choose $k = 0$ (this amounts to a translation of ϕ by an integer). The corresponding $h(n)$ are then exactly given by Example 3.1, resulting in $\phi = \chi_{[0,1]}$.

All these arguments prove the following proposition.

PROPOSITION 4.1. *The Haar basis* (1.9) *is the only orthonormal basis of compactly supported wavelets for which the associated averaging function* ϕ *has a symmetry axis.*

In the following subsection we explicitly characterize all the functions m_0 corresponding to orthonormal wavelet bases with compactly supported basic wavelet.

4.B. Characterization of all orthonormal, compactly supported wavelet bases. The basic condition (3.18) on the $h(n)$ can be rewritten as (see subsection 3A)

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1. \tag{4.7}$$

On the other hand, we have imposed, in Proposition 3.3, the following structure on m_0:

$$m_0(\xi) = \left[\tfrac{1}{2}(1 + e^{i\xi})\right]^N Q(e^{i\xi}), \tag{4.8}$$

where Q is a polynomial, since only finitely many $h(n)$ are non zero. Moreover, since all the $h(n)$ are real, all the coefficients in Q are real as well. From (4.8) we have

$$|m_0(\xi)|^2 = \left[\cos^2 \tfrac{1}{2}\xi\right]^N |Q(e^{i\xi})|^2.$$

Since $\overline{Q(e^{i\xi})} = Q(e^{-i\xi})$, the polynomial $|Q(e^{i\xi})|^2$ can be rewritten as a polynomial in $\cos \xi$, or, equivalently, as a polynomial in $\sin^2 \frac{1}{2}\xi$. Introducing the shorthand $y = \cos^2 \frac{1}{2}\xi$, (4.7) becomes

$$y^N P(1 - y) + (1 - y)^N P(y) = 1. \tag{4.9}$$

Any m_0 of type (4.8) which solves (4.7) corresponds therefore to a polynomial P solving (4.9) and satisfying

$$P(y) \geq 0 \quad \text{for} \quad y \in [0, 1]. \tag{4.10}$$

Conversely, every polynomial P satisfying both (4.9) and (4.10) leads to solutions of (4.7), with real coefficients $h(n)$.This is due to the following lemma of Riesz [27].

Lemma 4.2. *Let A be a positive trigonometric polynomial containing only cosines, $A(\xi) = \sum_{n=0}^{N} a_n \cos n\xi$ (with $a_n \in \mathbb{R}$). Then there exists a trigonometric polynomial B, of order N, $B(\xi) = \sum_{n=0}^{N} b_n e^{in\xi}$, with real coefficients b_n, such that*

$$|B(\xi)|^2 = A(\xi). \tag{4.11}$$

The proof of this lemma (see [27]) is simple and elegant. It constructs B explicitly; this construction is now widely used by engineers when designing filters. We include the proof here, because we shall come back to the construction later.

Proof: To

$$A(\xi) = a_0 + \frac{1}{2} \sum_{n=1}^{N} a_n (e^{in\xi} + e^{-in\xi})$$

$$= e^{-iN\xi} \left[\frac{1}{2} \sum_{n=0}^{N-1} a_{N-n} e^{in\xi} + a_0 e^{iN\xi} + \frac{1}{2} \sum_{n=1}^{N} a_n e^{i(N+n)\xi} \right]$$

we associate the polynomial

$$P_A(z) = \frac{1}{2}\sum_{n=0}^{N-1} a_{N-n} z^n + a_0 z^N + \frac{1}{2}\sum_{n=1}^{N} a_n z^{N+n}.$$

This polynomial has $2N$ zeros (counting multiplicity). Since $P_A(e^{i\xi}) = e^{iN\xi}A(\xi)$, it follows that the two polynomials $P_A(z)$ and $z^{2N}P_A(z^{-1})$ agree on the unit circle, and therefore on the whole complex plane. They have therefore the same zeros. This means that if z_0 is a zero of $P_A(z)$, $P_A(z_0) = 0$, then so is z_0^{-1}. On the other hand, since the a_n are real, $\overline{P_A(z)} = P_A(\bar{z})$. This implies that if z_0 is a zero of $P_A(z)$, then so is its complex conjugate $\bar{z}_0$. The zeros of $P_A(z)$ therefore come in quadruplets, z_0, $\bar{z}_0$, z_0^{-1} and $\bar{z}_0^{-1}$, or (if $z_0 = r_0$ is real) in duplets, $r_0, r_0^{-1} \in \mathbb{R}$. Let $z_j, \bar{z}_j, z_j^{-1}, \bar{z}_j^{-1}$ be the quadruplets of complex zeros of $P_A(z)$, and r_k, r_k^{-1} the real duplets,

$$P_A(z) = \frac{1}{2}a_N\left[\prod_{k=1}^{K}(z - r_k)(z - r_k^{-1})\right] \cdot \left[\prod_{j=1}^{J}(z - z_j)(z - \bar{z}_j)(z - z_j^{-1})(z - \bar{z}_j^{-1})\right].$$

For $z = e^{i\xi}$ on the unit circle, one finds

$$\left|(e^{i\xi} - z_0)(e^{i\xi} - \bar{z}_0^{-1})\right| = |z_0|^{-1}\left|(e^{i\xi} - z_0)(\bar{z}_0 - e^{-i\xi})\right| = |z_0|^{-1}|e^{i\xi} - z_0|^2.$$

Consequently,

$$\begin{aligned} A(\xi) &= |A(\xi)| = |P_A(e^{i\xi})| \\ &= \left[\frac{1}{2}|a_N|\prod_{k=1}^{K}|r_k|^{-1}\prod_{j=1}^{J}|z_j|^{-2}\right]\left|\prod_{k=1}^{K}(e^{i\xi} - r_k)\prod_{j=1}^{J}(e^{i\xi} - z_j)(e^{i\xi} - \bar{z}_j)\right|^2 \\ &= |B(\xi)|^2, \end{aligned}$$

where

$$B(\xi) = \left[\frac{1}{2}|a_N|\prod_{k=1}^{K}|r_k|^{-1}\prod_{j=1}^{J}|z_j|^{-2}\right]^{1/2} \cdot \prod_{k=1}^{K}(e^{i\xi} - r_k)\prod_{j=1}^{J}\left(e^{2i\xi} - 2e^{i\xi}\,\mathcal{Re}\, z_j + |z_j|^2\right) \tag{4.12}$$

is clearly a trigonometric polynomial of order N with only real coefficients.

Remarks. 1. Note that B is generally not unique. Out of any quadruplet of zeros z_0, $\bar{z}_0$, z_0^{-1}, $\bar{z}_0^{-1}$ one can choose the pair of zeros to retain, for the construction of B, in four different ways. For every duplet of real zeros of P_A two choices are possible. This results in 2^N different possibilities for B.

2. All these different possibilities, corresponding to different choices of the zeros of P_A to retain for B, constitute, however, the only solutions to (4.11). One can show (see [27]) that, up to an arbitrary phase factor $\pm e^{iK\xi}$, $K \in \mathbf{Z}$, *all* the polynomials B satisfying (4.11) are necessarily of the form (4.12).

If P is a polynomial satisfying (4.9) and (4.10), then Lemma 4.2 tells us that there exists a trigonometric polynomial of the same order such that

$$|Q(e^{i\xi})|^2 = P(\sin^2 \tfrac{1}{2}\xi) = P\left(\tfrac{1}{2}(1 - \cos \xi)\right).$$

It follows that $m_0(\xi) = [\frac{1}{2}(1 + e^{i\xi})]^N Q(e^{i\xi})$ satisfies (4.7). If, moreover,

$$\sup_{\xi} |Q(e^{i\xi})| = \sup_{y \in [0,1]} |P(y)|^{1/2} < 2^{N-1},$$

then all the conditions of Theorem 3.6 are satisfied, and there exists an associated orthonormal wavelet basis.

To construct compactly supported orthonormal wavelet bases, with m_0 of type (4.8), it is therefore necessary and sufficient to find polynomials P solving (4.9) and (4.10), which are moreover strictly bounded above by $2^{2(N-1)}$.

The following two combinatorial lemmas allow one to "guess" a particular solution of (4.9).

LEMMA 4.3.

$$\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}.$$

Proof: Define $S_{n,k} = \sum_{j=0}^{k} \binom{n+j}{j}$. Then

$$S_{n+1,k+1} = \frac{(k+n+2)!}{(k+1)!(n+1)!} + \sum_{j=0}^{k} \frac{(n+j)!}{(n+1)!j!}(n+j+1)$$

$$= \binom{k+n+2}{k+1} + S_{n,k} + \sum_{j=1}^{k} \frac{(n+j)!}{(n+1)!(j-1)!}$$

$$= \binom{k+n+2}{n+1} + S_{n,k} + \left[S_{n+1,k+1} - \binom{k+n+2}{k+1} - \binom{k+n+1}{k}\right].$$

Hence

$$S_{n,k} = \binom{n+k+1}{k}.$$

LEMMA 4.4.

$$\sum_{j=0}^{n} \binom{n+j}{j} \left[y^j (1-y)^{n+1} + y^{n+1}(1-y)^j \right] = 1.$$

Proof: Define $A_{n,j} = \binom{n+j}{j}$. Then, by Lemma 4.3, $\sum_{j=0}^{k} A_{n,j} = A_{n+1,k}$. Define

$$S_n(y) = \sum_{j=0}^{n} \binom{n+j}{j} \left[y^j (1-y)^{n+1} + y^{n+1}(1-y)^j \right].$$

Clearly,

$$S_0(a) = (1-a) + a = 1.$$

We shall prove that $S_n(a) = S_{n-1}(a)$, which proves the lemma. By repeatedly inserting factors $[(1-a)+a] = 1$, we find

$$\begin{aligned}
S_{n-1}(a) &= \sum_{j=0}^{n-1} A_{n-1,j}\left[(1-a)^n a^j + a^n (1-a)^j\right] \\
&= A_{n-1,0}\left[(1-a)^{n+1} + a^{n+1}\right] \\
&\quad + \left(A_{n-1,0} + A_{n-1,1}\right)\left[(1-a)^n a + a^n (1-a)\right] \\
&\quad + \sum_{j=2}^{n-1} A_{n-1,j}\left[(1-a)^n a^j + a^n (1-a)^j\right] \\
&= \cdots \\
&= \sum_{j=0}^{n-1}\left[\sum_{k=0}^{j} A_{n-1,k}\right]\left[(1-a)^{n+1} a^j + a^{n+1}(1-a)^j\right] \\
&\quad + 2\left[\sum_{k=0}^{n-1} A_{n-1,k}\right](1-a)^n a^n \\
&= \sum_{j=0}^{n-1} A_{n,j}\left[(1-a)^{n+1} a^j + a^{n+1}(1-a)^j\right] \\
&\quad + 2A_{n,n-1}\left[(1-a)^{n+1} a^n + a^{n+1}(1-a)^n\right] \\
&= S_n(a) \qquad \text{(since } 2A_{n,n-1} = A_{n,n}\text{)}.
\end{aligned}$$

It follows that the polynomial of order $N - 1$,

$$P_N(y) = \sum_{j=0}^{N-1} \binom{N-1+j}{j} y^j, \tag{4.13}$$

solves (4.9). Since all the coefficients in this polynomial are positive, (4.10) is clearly also satisfied.

The two explicit examples of compactly supported wavelet bases we have seen so far, i.e., Example 3.1 and (3.42), correspond exactly to a polynomial of type (4.13), with $N = 1, 2$, respectively. For Example 3.1 one has $m_0(\xi) = \frac{1}{2}(1 + e^{i\xi})$, i.e., $N = 1$, and $Q(e^{i\xi}) = 1$, hence $P(y) = 1 = P_1(y)$. For the second example (3.42), we find (see (4.4)) $m_0(\xi) = [\frac{1}{2}(1 + e^{i\xi})]^2 \frac{1}{2}[(1 \mp \sqrt{3})e^{i\xi}]$, corresponding to $N = 2$ and $|Q(e^{i\xi})|^2 = 2 - \cos\xi = 1 + 2\sin^2\frac{1}{2}\xi$; hence $P(y) = 1 + 2y = P_2(y)$.

In fact, for given N, P_N is the *only* polynomial of order less than N which solves (4.9). Even more is true: for *any* polynomial P solving (4.9), the first N terms (orders 0 up till $N - 1$) are exactly given by P_N. This is because (4.9) completely determines the first N coefficients $p_0, \cdots, p_{N-1}$ in $P(y) = \sum_{n=0}^{K} p_n y^n$. Since the first term in (4.9) is already of order N, only the second term plays a role in the cancellations for y^k, $k = 0, \cdots, N - 1$. This leads to

$$\begin{aligned} p_0 &= 1, \\ p_k &= \sum_{n=0}^{k-1} (-1)^{k-n+1} \binom{N}{k-n} p_n, \qquad k = 1, \cdots, N-1, \end{aligned} \tag{4.14}$$

from which the p_k, $k = 1, \cdots, N - 1$, can be determined recursively. Since P_N solves (4.9), it follows from (4.13) that

$$p_k = \binom{N+k-1}{k}.$$

Consequently, *any* polynomial P solving (4.9) is of the form

$$P(y) = P_N(y) + y^N R(y). \tag{4.15}$$

Substitution of (4.15) into (4.9) leads to the following equation for the polynomial R:

$$y^N(1-y)^N R(1-y) + (1-y)^N y^N R(y) = 0,$$

or

$$R(1-y) + R(y) = 0.$$

The polynomial R is therefore antisymmetric with respect to $y = \frac{1}{2}$, or

$$R(y) = \tilde{R}(\tfrac{1}{2} - y),$$

where $\tilde{R}$ is an odd polynomial.

To summarize, we have the following explicit characterization of all solutions m_0 of (4.7), corresponding to only finitely many non-zero $h(n)$.

PROPOSITION 4.5. *Any trigonometric polynomial solution m_0 of (4.7) is of the form*

$$m_0(\xi) = \left[\tfrac{1}{2}(1 + e^{i\xi})\right]^N Q(e^{i\xi}), \tag{4.16}$$

where $N \in \mathbb{N}$, $N \geq 1$, and where Q is a polynomial such that

$$\left|Q(e^{i\xi})\right|^2 = \sum_{k=0}^{N-1} \binom{N-1+k}{k} \sin^{2k} \tfrac{1}{2}\xi + \left[\sin^{2N}\tfrac{1}{2}\xi\right] R\left(\tfrac{1}{2}\cos \xi\right), \tag{4.17}$$

where R is an odd polynomial.

Remarks. 1. Since the proof of Lemma 4.2 shows explicitly how to construct all possible polynomials Q once $|Q(e^{i\xi})|^2$ is known, this proposition is indeed an explicit characterization of all the solutions m_0 of (4.7).

2. In constructing m_0, there are therefore 3 steps at which choices can be made,

(i) choosing $N \in \mathbb{N} \setminus \{0\}$,
(ii) choosing an odd polynomial R (with some restrictions),
(iii) choosing pairs of zeros out of each quadruplet of complex zeros, and one zero out of each duplet of real zeros, of $P_N(z) + z^N R(z - \frac{1}{2})$ (see the proof of Lemma 4.2).

The odd polynomial R cannot be chosen completely freely. One needs, of course, the fact that

$$P_N(y) + y^N R\left(\tfrac{1}{2} - y\right) \geqq 0 \quad \text{for} \quad 0 \leqq y \leqq 1. \tag{4.18}$$

Moreover, condition (v) in Theorem 3.6 requires that

$$\sup_{0 \leqq y \leqq 1} \left[P_N(y) + y^N R\left(\tfrac{1}{2} - y\right)\right] < 2^{2(N-1)}. \tag{4.19}$$

3. For $N = 1$, (4.16), (4.17) and (4.18) reduce to

$$m_0(\xi) = \tfrac{1}{2}(1 + e^{i\xi}) Q(e^{i\xi}) \tag{4.20}$$

with

$$\left|Q(e^{i\xi}\right|^2 = 1 + \sin^2 \tfrac{1}{2}\xi R\left(\tfrac{1}{2}\cos \xi\right), \tag{4.21}$$

where R is an odd polynomial such that

$$-\frac{2}{1 - 2|x|} \leqq R(x) \leqq \frac{2}{1 + 2|x|} \quad \text{for} \quad |x| \leqq \tfrac{1}{2}.$$

These conditions can already be found in the construction of conjugate quadrature mirror filters in [24]. The condition (4.19) is impossible to satisfy, however, because $P_1(0) = 1$.

4. Using a different method, Y. Meyer constructs in [28] another polynomial solving (4.7). The solutions to (4.7) proposed in [28] are

$$|m_0(\xi)|^2 = 1 - \frac{(2N-1)!}{[(N-1)!]^2 2^{2N-1}} \int_0^\xi \sin^{2N-1} x \, dx. \tag{4.22}$$

This is clearly an even trigonometric polynomial of order $2N - 1$. It turns out to be divisible by $(\frac{1}{2}(1 + \cos \xi))^N = (\cos^2 \frac{1}{2}\xi)^N$. Therefore, by Proposition 4.5, (4.22) is exactly equal to

$$(\cos^2 \tfrac{1}{2}\xi)^N P_N(\sin^2 \tfrac{1}{2}\xi).$$

4.C. A family of examples with arbitrarily high regularity. In the remainder of this section, we shall concern ourselves with a special family of functions m_0, and the corresponding wavelet bases. We follow the prescriptions of Remark 2 after Proposition 4.5. For every $N \in \mathbf{N}$, $N \geq 1$, we choose Q of minimal order, i.e., $R \equiv 0$, $|Q(e^{i\xi})|^2 = P_N(\sin^2 \frac{1}{2}\xi)$. This choice satisfies both the conditions (4.18) and (4.19). From (4.13) the positivity of $P_N(y)$ for $0 \leq y \leq 1$ is immediate. Since P_N is strictly increasing for $y \geq 0$, it follows that

$$\begin{aligned} \sup_{y \in [0,1]} P_N(y) = P_N(1) &= \binom{2N-1}{N-1} = \frac{1}{2}\left[\binom{2N-1}{N-1} + \binom{2N-1}{N}\right] \\ &< \frac{1}{2} \sum_{k=0}^{2N-1} \binom{2N-1}{k} = 2^{2(N-1)}, \end{aligned} \tag{4.23}$$

where we have used Lemma 4.3 in the second equality. This fixes $|Q|^2$. In the construction (via Lemma 4.2) of Q from $|Q|^2$, we systematically retain all the zeros inside the unit circle (this corresponds to a "minimal phase" choice in filter design). For $N \in \mathbf{N}$, $N > 1$ fixed, this determines Q unambiguously, up to a phase factor $e^{iK\xi}$, $K \in \mathbf{Z}$. For the sake of definiteness we fix this phase factor so that Q contains only positive frequencies, starting from zero, i.e.,

$$Q_N(e^{i\xi}) = \sum_{n=0}^{N-1} q_N(n) e^{in\xi} \quad \text{with} \quad q_0 \neq 0. \tag{4.24}$$

These choices uniquely determine Q_N. We shall denote the corresponding m_0 by ${}_N m_0$,

$$\begin{aligned} {}_N m_0(\xi) &= \left[\tfrac{1}{2}(1 + e^{i\xi})\right]^N \sum_{n=0}^{N-1} q_N(n) e^{in\xi} \\ &= 2^{-1/2} \sum_{n=0}^{2N-1} h_N(n) e^{in\xi}. \end{aligned}$$

Table 1 lists the coefficients $h_N(n)$ for the cases $N = 2, 3, \cdots, 10$. For the lowest values of N, $Q_N(\xi)$ can be determined analytically. One has, e.g.,

$$Q_2(\xi) = \tfrac{1}{2}\left[(1+\sqrt{3}) + (1-\sqrt{3})e^{i\xi}\right] \qquad \text{(see (4.4))}$$

and

$$Q_3(\xi) = \tfrac{1}{4}\Big[\Big(1+\sqrt{10}+\sqrt{5+2\sqrt{10}}\Big) + 2(1-\sqrt{10})e^{i\xi} + \Big(1+\sqrt{10}-\sqrt{5+2\sqrt{10}}\Big)e^{2i\xi}\Big].$$

For larger values of N, the coefficients in Table 1 were computed numerically.

Since the ${}_N m_0$ satisfy all the conditions of Theorem 3.6, there exists an associated orthonormal basis of continuous wavelets with compact support for every ${}_N m_0$. Let us denote the corresponding ϕ, ψ functions by ${}_N\phi$, ${}_N\psi$. Since $h_N(n) = 0$ for $n < 0$ and $n > 2N-1$, it follows (see the discussion at the start of Section 4) that $\operatorname{supp}({}_N\phi) = [0, 2N-1]$. The support of ${}_N\psi$,

$$({}_N\psi)(x) = \sum_{n=0}^{2N-1} (-1)^n h_N(-n+1)\, {}_N\phi(2x-n),$$

is therefore given by $[-(N-1), N]$. Note that an additional phase factor $e^{iK\xi}$, $K \in \mathbf{Z}$, in (4.24) would amount to shifting the $h_N(n)$ by K, i.e., to shifting the function ${}_N\phi$ by an integer, which does not affect the multiresolution analysis construction. The wavelet ${}_N\psi$ is unaffected by this shift.

From Theorem 3.6, we know that ${}_N\phi$ and ${}_N\psi$ are bounded, continuous functions. For large N, ${}_N\phi$ and ${}_N\psi$ are, in fact, much more regular. To see this, we shall need the following generalization of Lemma 3.2.

LEMMA 4.6. *If* $m_0(\xi) = [\tfrac{1}{2}(1+e^{i\xi})]^N \mathscr{F}(\xi)$, *where* $\mathscr{F}(\xi) = \sum_n f_n e^{in\xi}$ *satisfies*

(4.25) $$\sum_n |f_n|\,|n|^{\varepsilon} < \infty \quad \textit{for some} \quad \varepsilon > 0,$$

(4.26) $$\sup_\xi \left|\mathscr{F}(\xi)\mathscr{F}(\tfrac{1}{2}\xi)\cdots\mathscr{F}(2^{-k+1}\xi)\right| = B_k,$$

then

(4.27) $$\left|\prod_{j=1}^{\infty} m_0(2^{-j}\xi)\right| \le C(1+|\xi|)^{-N+\log B_k/(k\log 2)}.$$

Proof: Define

$$\mathscr{F}_k(\xi) = \prod_{j=0}^{k} \mathscr{F}(2^{-j}\xi).$$

Table 1. The coefficients $h_N(n)$ $(n = 0, \cdots, 2N-1)$ for $N = 2, 3, \cdots, 10$.

	n	$h_N(n)$		n	$h_N(n)$
$N = 2$	0	.482962913145	$N = 8$	0	.054415842243
	1	.836516303738		1	.312871590914
	2	.224143868042		2	.675630736297
	3	−.129409522551		3	.585354683654
$N = 3$	0	.332670552950		4	−.015829105256
	1	.806891509311		5	−.284015542962
	2	.459877502118		6	.000472484574
	3	−.135011020010		7	.128747426620
	4	−.085441273882		8	−.017369301002
	5	.035226291882		9	−.044088253931
$N = 4$	0	.230377813309		10	.013981027917
	1	.714846570553		11	.008746094047
	2	.630880767930		12	−.004870352993
	3	−.027983769417		13	−.000391740373
	4	−.187034811719		14	.000675449406
	5	.030841381836		15	−.000117476784
	6	.032883011667	$N = 9$	0	.038077947364
	7	−.010597401785		1	.243834674613
$N = 5$	0	.160102397974		2	.604823123690
	1	.603829269797		3	.657288078051
	2	.724308528438		4	.133197385825
	3	.138428145901		5	−.293273783279
	4	−.242294887066		6	−.096840783223
	5	−.032244869585		7	.148540749338
	6	.077571493840		8	.030725681479
	7	−.006241490213		9	−.067632829061
	8	−.012580751999		10	.000250947115
	9	.003335725285		11	.022361662124
$N = 6$	0	.111540743350		12	−.004723204758
	1	.494623890398		13	−.004281503682
	2	.751133908021		14	.001847646883
	3	.315250351709		15	.000230385764
	4	−.226264693965		16	−.000251963189
	5	−.129766867567		17	.000039347320
	6	.097501605587	$N = 10$	0	.026670057901
	7	.027522865530		1	.188176800078
	8	−.031582039318		2	.527201188932
	9	.000553842201		3	.688459039454
	10	.004777257511		4	.281172343661
	11	−.001077301085		5	−.249846424327
$N = 7$	0	.077852054085		6	−.195946274377
	1	.396539319482		7	.127369340336
	2	.729132090846		8	.093057364604
	3	.469782287405		9	−.071394147166
	4	−.143906003929		10	−.029457536822
	5	−.224036184994		11	.033212674059
	6	.071309219267		12	.003606553567
	7	.080612609151		13	−.010733175483
	8	−.038029936935		14	.001395351747
	9	−.016574541631		15	.001992405295
	10	.012550998556		16	−.000685856695
	11	.000429577973		17	−.000116466855
	12	−.001801640704		18	.000093588670
	13	.000353713800		19	−.000013264203

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$$\prod_{j=1}^{\infty} \mathscr{F}(2^{-j}\xi) = \prod_{j=0}^{\infty} \mathscr{F}_k\left[(2^k)^{-j}\tfrac{1}{2}\xi\right].$$

Repeating the proof of Lemma 3.2, with multiplication factor 2^k instead of 2 leads to

$$\left|\prod_{j=0}^{\infty} \mathscr{F}_k\left[(2^k)^{-j}\xi\right]\right| \leq C\exp\left\{\log B_k \log|\xi|/\log(2^k)\right\}.$$

This implies (4.27).

To interpolate between the standard spaces C^k of k times continuously differentiable functions, we shall use, for $\alpha \notin \mathbb{N}$, $\alpha > 0$, the spaces defined by

$$f \in C^\alpha \Leftrightarrow \int dx\, |\hat{f}(\xi)|\,(1 + |\xi|)^{1+\alpha} < \infty. \tag{4.28}$$

Note that, for $\alpha = k \in \mathbb{N}$, the condition (4.28) implies $f \in C^k$, but is not necessary.

We then have the following

PROPOSITION 4.7. *There exists $\lambda > 0$ such that, for all $N \in \mathbb{N}$, $N \geq 2$,*

$$_N\phi,\, _N\psi \in C^{\lambda N}. \tag{4.29}$$

Proof: We shall apply Lemma 4.6. Since $Q_N(e^{i\xi})$ has only a finite number of terms, (4.25) is obviously satisfied. We compute

$$B_2 = \sup\left|Q_N(e^{i\xi})Q_N(e^{i\xi/2})\right| = \sup_{\xi}\left|P_N(\sin^2\tfrac{1}{2}\xi)\,P_N(\sin^2\tfrac{1}{4}\xi)\right|$$

$$= \sup_{0\leq y\leq 1}\left|P_N(4y(1-y))P_N(y)\right|.$$

First, note that (see (4.28))

$$\sup_{0\leq y\leq 1} P_N(y) = P_N(1) < 2^{2(N-1)}.$$

Secondly,

$$P_N(y) = \sum_{k=0}^{N-1}\binom{N+k-1}{k}y^k$$

$$\leq \sum_{k=0}^{N-1} 2^{N+k-1}y^k \leq 2^{N-1}N\max\left(1,(2y)^N\right).$$

Hence, for $y \leqq \frac{1}{2}$,

$$P_N(y)P_N(4y(1-y)) \leqq N2^{N-1}2^{2(N-1)} = N2^{3(N-1)}.$$

For $y \geqq \frac{1}{4}(2+\sqrt{2})$, or $4y(1-y) \leqq \frac{1}{2}$,

$$P_N(y)P_N(4y(1-y)) \leqq 2^{2(N-1)}N2^{N-1} = N2^{3(N-1)}.$$

Finally, for $\frac{1}{2} \leqq y \leqq \frac{1}{4}(2+\sqrt{2})$,

$$P_N(y)P_N(4y(1-y)) \leqq N^2 2^{4N-2}\Big(\sup_{0\leq y\leq 1}\big[4y^2(1-y)\big]\Big)^N$$

$$= N^2 2^{4N-2}\left(\tfrac{16}{27}\right)^N,$$

or

$$B_2 \leqq N2^{2N-1}\left(\tfrac{16}{27}\right)^{N/2}.$$

Consequently,

$$\left|({}_N\phi)^\wedge(\xi)\right| = (2\pi)^{-1/2}\left|\prod_{j=1}^{\infty} m_0(2^{-j}\xi)\right|$$

$$\leqq C(1+|\xi|)^{[\log N - N\log(3\sqrt{3}/4)]/2\log 2}.$$

This exponent is smaller than -1 for $N \geqq 16$. For smaller values of N, one can use the explicit estimate

$$B_1 = \left[\binom{2N-1}{N}\right]^{1/2}$$

to prove that

$$\left|({}_N\phi)^\wedge(\xi)\right| \leqq C(1+|\xi|)^{-1-\kappa N}$$

for some $\kappa > 0$, for all $N \leqq 16$. Hence (4.29) holds for ${}_N\phi$, for some $\lambda > 0$, and for all $N \geqq 2$. Since ${}_N\psi$ is always a finite linear combination of translated and dilated versions of ${}_N\phi$, the same holds for ${}_N\psi$.

Remarks. 1. Since $|\text{supp}({}_N\phi)| = |\text{supp}({}_N\psi)| = 2N-1$, (4.29) shows that the regularity of ${}_N\phi$, ${}_N\psi$ increases linearly with their support width, as announced in the introduction. It turns out that linear increase of the support width with the regularity of ϕ, ψ is the best one can do. More precisely, if a C^k-function ϕ satisfies an equation of the type

$$\phi(x) = \sum_{n=0}^{N} c_n\phi(2x-n)$$

(without necessarily being connected to multiresolution analysis), and if supp $\phi \subset [0, N]$, then $k \leq N - 2$. For a proof, see [30].

2. The estimate for λ obtained in this proof is, of course, not very good; the argument is too simple. Asymptotically, for large N, one finds

$$_N\phi, {}_N\psi \in C^{(\mu-\epsilon)N}$$

with

$$\mu \sim \frac{\log\left(\frac{3}{4}\sqrt{3}\right)}{2\log 2} + O\left(\frac{\log N}{N}\right) \gtrsim .1887 + O\left(\frac{\log N}{N}\right).$$

The same technique, with a little more work, leads to slightly better estimates if larger values of k are used. Using $k = 4$, e.g., leads to

$$\mu \gtrsim .1936 + O\left(N^{-1}\log N\right).$$

Since the map $y \mapsto 4y(1-y)$ has a fixed point, at $y = \frac{3}{4}$, one finds

$$B_k \geq \left[P_N\left(\tfrac{3}{4}\right)\right]^{k/2}.$$

One can show that

$$P_N\left(\tfrac{3}{4}\right) \sim C3^N.$$

Even for arbitrarily large k, the values of μ obtained by this method are therefore limited by

$$\mu \leq 1 - \frac{\log 3}{2\log 2} + O\left(N^{-1}\log N\right) \cong .2075 + O\left(N^{-1}\log N\right).$$

3. Using a more sophisticated method than the brutal estimates above, Y. Meyer [28] showed that, again asymptotically for large N,

$$_N\phi, {}_N\psi \in C^{(\mu-\epsilon)N}$$

with $\mu = \log(4/\pi)/\log 2 \cong .3485$. His proof uses (4.22) rather than P_N.

4. For small values of N, better estimates can be obtained for the regularity of the $_N\phi$, $_N\psi$ by yet a third method. This method is based on a generalization of a technique used by Riesz in the proof that "Riesz products" can lead to continuous, nowhere differentiable functions. I would like to thank Y. Meyer for introducing me to this technique, and for showing me how to use it to prove $_2\phi, {}_2\psi \in C^{.5-\epsilon}$. The proof, and a generalization for $N \geq 3$, are given in the Appendix. It works very well for small values of N, but does not, however, give good asymptotic results. For large N, it leads to logarithmic rather than linear increase of the regularity of the $_N\phi$, $_N\psi$.

Table 2. Regularity estimates.
For $N = 2, \cdots, 10$, we give α_N so that ${}_N\phi, {}_N\psi \in C^{\alpha_N}$

N	α_N
2	$.5 - \varepsilon$
3	.915
4	1.275
5	1.596
6	1.888
7	2.158
8	2.415
9	2.661
10	2.902

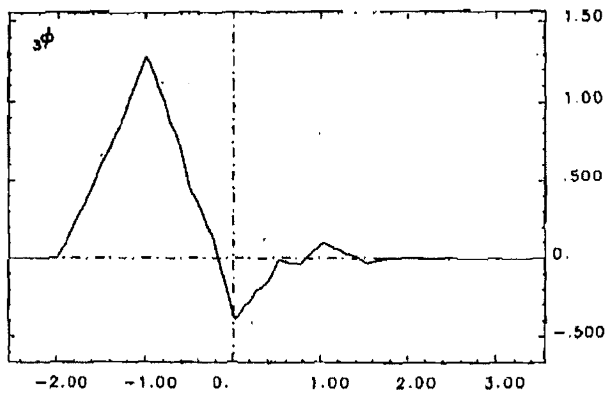
To conclude this paper, we give in Figure 7 the graphs of ${}_N\phi$, ${}_N\psi$ and their Fourier transforms $({}_N\phi)^\wedge$, $({}_N\psi)^\wedge$, for $N = 3, 5, 7, 9$. (For $N = 2$, these graphs were given in Figure 5.) The graphs were plotted by means of the "graphical algorithm" explained in subsection 2B, using the coefficients $h_N(n)$ of Table 1. One clearly sees that the ${}_N\phi$, ${}_N\psi$ become more regular as N increases. Also noticeable is that $|({}_N\phi)^\wedge|$, $|({}_N\psi)^\wedge|$ become "flatter" as N increases, around 0 and $2\pi \cong 6.28$. This is a direct consequence of (4.7) and (4.8). By (4.3), $({}_Nm_0)(\xi)$ has a zero of order N at $\xi = \pi$. It follows that, by (4.7), $({}_Nm_0)(0) = 1$, and that the first $N - 1$ derivatives $({}_Nm_0)^{(j)}(\xi)$ of ${}_Nm_0$ are zero in $\xi = 0$. Since (this follows from (3.45)) $({}_N\psi)^\wedge(\xi) = \overline{{}_Nm_0(\pi + \frac{1}{2}\xi)}\,({}_N\phi)^\wedge(\frac{1}{2}\xi)$, this means that $[({}_N\psi)^\wedge]^{(k)}(0) = 0$ for $k = 0, \cdots, N - 1$, or $\int dx\, x^k ({}_N\psi)(x) = 0$ for $k = 0, \cdots, N - 1$. The present construction leads thus also to orthonormal bases of compactly supported wavelets with an arbitrarily high number of zero moments. This property could be useful for quantum field theory (see [18]).

It is also quite striking that the "effective support" (where $|({}_N\psi)(x)| \geq .01\|{}_N\psi\|_\infty$, say) of ${}_N\psi$ is quite a bit smaller than its total support, for N not too small. This is due to the very small value of the $h_N(n)$ for large n (see Table 1). Table 2 lists the estimates for the "regularity index" α_N (where ${}_N\phi$, ${}_N\psi \in C^{\alpha_N}$) for $N = 2, 3, \cdots, 10$, computed using the method explained in the Appendix.

Remark. Using a different approach (see [30]), these estimates for the regularity index α_N can be sharpened. For $N = 2$ one finds, e.g., $\alpha_2 = 2 - \ln(1 + \sqrt{3})/\ln 2 \simeq .550\cdots$. This is the best possible exponent for $N = 2$ (see [30]).

Appendix
Sharper Regularity Estimates for ${}_N\phi$, ${}_N\psi$

The estimates given here are based on a different way of calculating (4.28). Using the facts that $|({}_N\phi)^\wedge(\xi)| = (2\pi)^{-1/2}\prod_{j=1}^{\infty}|({}_Nm_0)(2^{-j}\xi)|$ is even (because ${}_Nm_0$ is a trigonometric polynomial with real coefficients) and that $|({}_Nm_0)(\xi)| \leq 1$



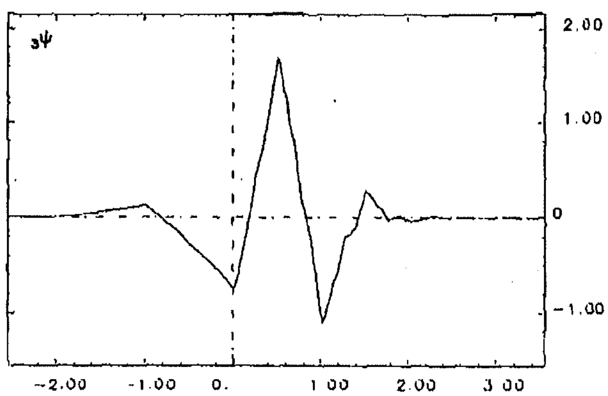


Figure 7. The functions ${}_N\phi$, ${}_N\psi$ and the modulus of their Fourier transforms $|({}_N\phi)^\wedge|$, $|({}_N\psi)^\wedge|$, for increasing values of N (see text). We have each time shifted ${}_N\phi$ by $N-1$, so that $\text{supp}({}_N\phi) = \text{supp}({}_N\psi) = [-(N-1), N]$. One clearly sees that the ${}_N\phi$, ${}_N\psi$ become more regular as N increases. The function ${}_N\phi$ has been plotted using the "graphical construction algorithm" explained in subsection 2B, with the weighting coefficients ${}_Nh(n)$ given in Table 1. Only 7 iterations were needed. The plot of ${}_N\psi$ then follows from $({}_N\psi)(x) = \sqrt{2}\Sigma_n(-1)^n h_N(-n+1)({}_N\phi)(2x-n)$.

(see (4.7)), we find

$$\int d\xi\,|({}_N\phi)^\wedge(\xi)|(1+|\xi|)^{1+\alpha} \le (2\pi)^{-1/2}2^{\alpha+1}(1+a)^{\alpha+1} \cdot \left\{ a + \sum_{m=0}^{\infty} 2^{(m+1)(\alpha+1)} \int_{2^m a}^{2^{m+1}a} d\xi \prod_{j=0}^{m} |({}_N m_0)(2^{-j}\xi)| \right\}, \tag{A.1}$$

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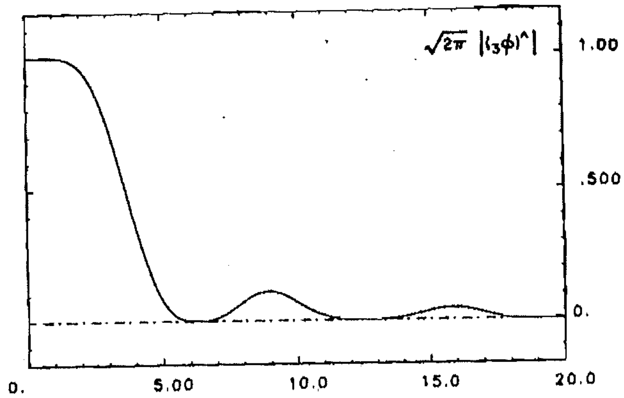

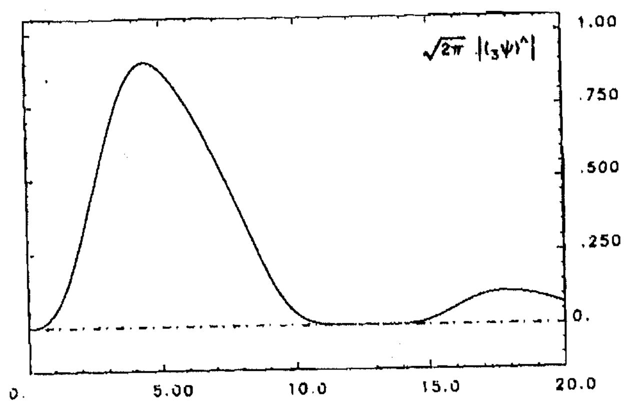

Figure 7. For the plots of $|({}_N\phi)|^\wedge$ the infinite product (3.46) was computed (truncated at $j = 10$), $(2\pi)^{1/2}|({}_N\phi)^\wedge(\xi)| = \prod_{j=1}^{\infty}|{}_N m_0(2^{-j}\xi)|$, with ${}_N m_0(\xi) = 2^{-1/2}\Sigma_n h_N(n)e^{in\xi}$, where the $h_N(n)$ are given in Table 1. The plot of $|({}_N\psi)^\wedge(\xi)|$ then follows from

$$|({}_N\psi)^\wedge(\xi)| = 2^{-1/2}|\Sigma_n(-1)^n h(-n+1)e^{in\xi/2}|\,|({}_N\phi)^\wedge(\tfrac{1}{2}\xi)|.$$

where $a > 0$ is arbitrary for the moment. Using $({}_N m_0)(\xi) = [\frac{1}{2}(1 + e^{i\xi})]^N Q_N(e^{i\xi})$, we find

$$\prod_{j=0}^{m}\left|({}_N m_0)(2^{-j}\xi)\right| = \left[\frac{|\sin\xi|}{2^{m+1}|\sin(2^{-m-1}\xi)|}\right]^N \prod_{j=0}^{m}\left|Q_N\left(e^{-2^{-j}\xi}\right)\right|.$$

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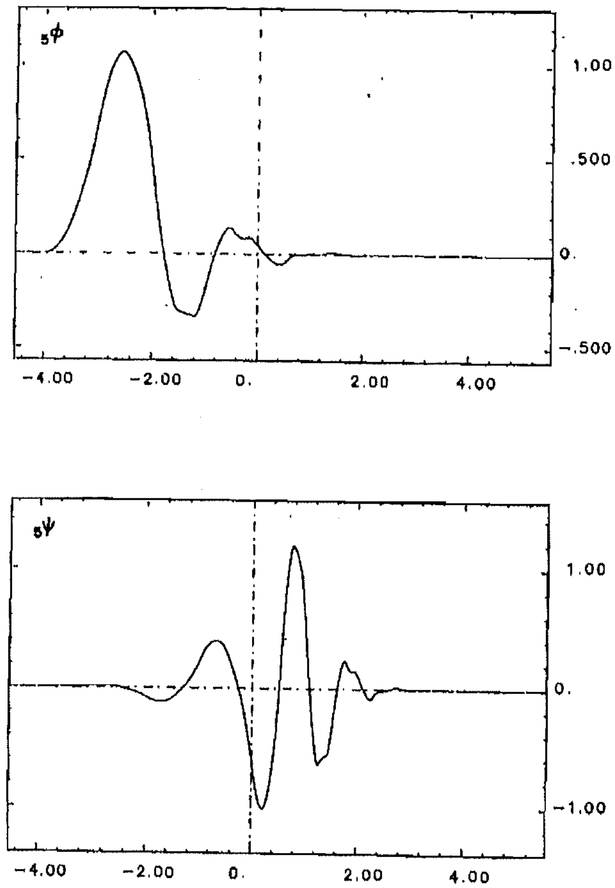

Figure 7. Continued

If we choose $a = \frac{2}{3}\pi$, then $|\sin(2^{-m-1}\xi)| \geq \sqrt{\frac{3}{2}}$ for $2^m a \leq |\xi| \leq 2^{m+1}a$. Hence

$$\int d\xi \left|({}_N\phi)^\wedge(\xi)\right|(1+|\xi|)^{1+\alpha}$$

$$\leq C_1 + C_2 \sum_{m=0}^{\infty} 2^{(\alpha+1)m}\, 2^{-N(m+1)} \int_{2^m 2\pi/3}^{2^{m+1}2\pi/3} \prod_{j=0}^{m} \left|\mathcal{Q}_N\left(e^{i2^{-j}\xi}\right)\right|$$

$$\leq C_1 + C_2 \sum_{m=0}^{\infty} 2^{(\alpha+1)m}\, 2^{-N(m+1)} (2^{m+1}\pi)^{1/2} \left[\int_0^{2^{m+1}\pi} d\xi \prod_{j=0}^{m} \left|\mathcal{Q}_N\left(e^{i2^{-j}\xi}\right)\right|^2\right]^{1/2}$$

$$\leq C_1 + C_3 \sum_{m=0}^{\infty} 2^{m(\alpha-N+1)} \left[\int_0^{2\pi} d\xi \prod_{j=0}^{m} P_N\left(\sin^2(2^j \tfrac{1}{2}\xi)\right)\right]^{1/2},$$

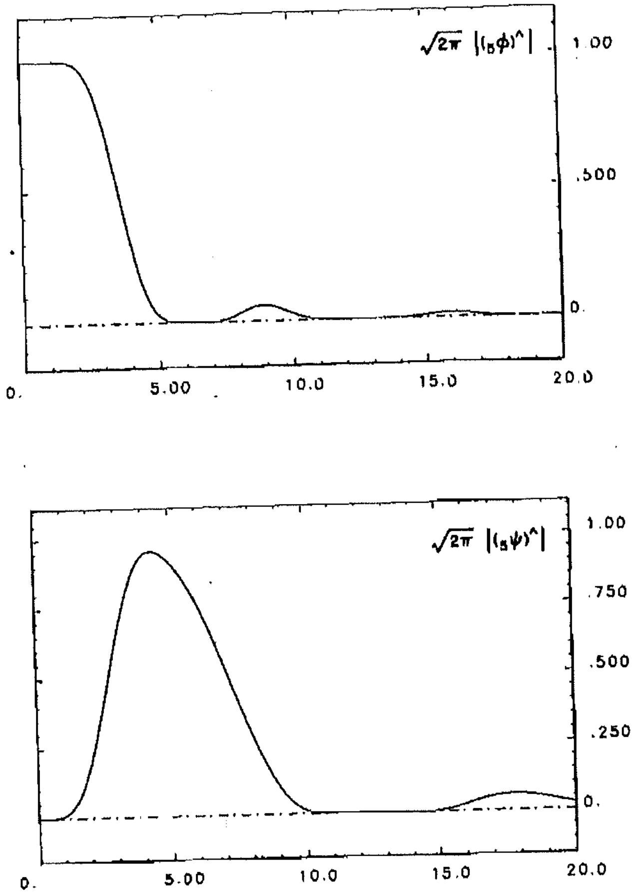

Figure 7. Continued

where we have used $|Q_N(e^{i\xi})|^2 = P_N(\sin^2\frac{1}{2}\xi)$ (see subsection 4C). It follows that (A.1) is convergent, hence ${}_N\phi,\ {}_N\psi \in C^\alpha$, if

$$\text{(A.2)}\quad \limsup_{m\to\infty}\,(2m\log 2)^{-1}\log\left[\int_0^{2\pi} d\xi \prod_0^{2\pi} P_N\left(\sin^2(2^{j-1}\xi)\right)\right] \leq N-1-\alpha.$$

We know P_N explicitly (see (4.13)),

$$\text{(A.3)}\qquad P_N\left(\sin^2\tfrac{1}{2}\xi\right) = \sum_{k=0}^{N-1}\binom{N+k-1}{k}\left(\sin^2\tfrac{1}{2}\xi\right)^k.$$

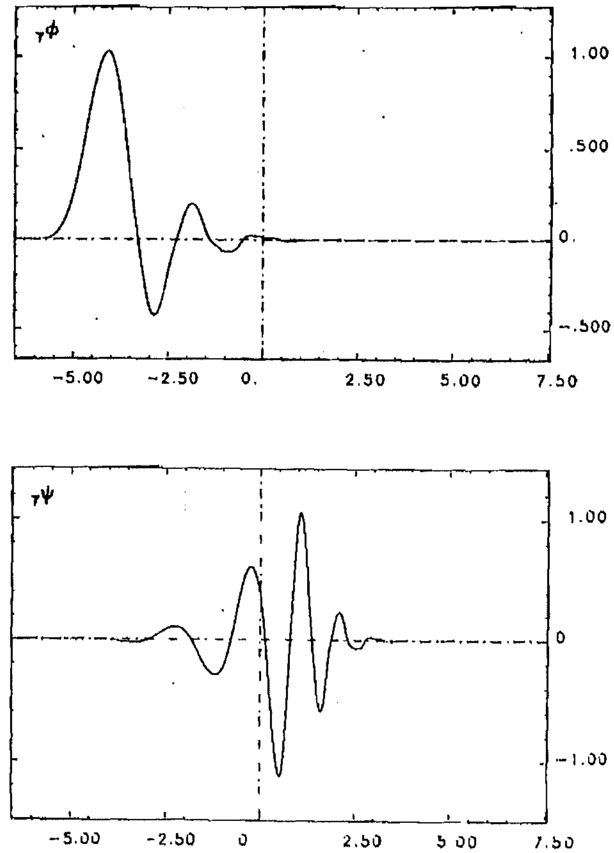

Figure 7. Continued

This can be rewritten as

$$P_N(\sin^2\tfrac{1}{2}\xi) = \sum_{l=-(N-1)}^{N-1} a_{N,l}\, e^{il\xi}, \tag{A.4}$$

where the $a_{N,l}$ are symmetric, $a_{N,l} = a_{N,-l}$, and can be calculated explicitly from (A.3). The product $\prod_{j=0}^{m} P_N(\sin^2(2^{j-1}\xi))$ is therefore a symmetric trigonometric polynomial of order $2^m(N-1)$,

$$\prod_{j=0}^{m} P_N\big(\sin^2(2^{j-1}\xi)\big) = \sum_{l=-(N-1)2^m}^{(N-1)2^m} J_{N,m;k}\, e^{ik\xi}. \tag{A.5}$$

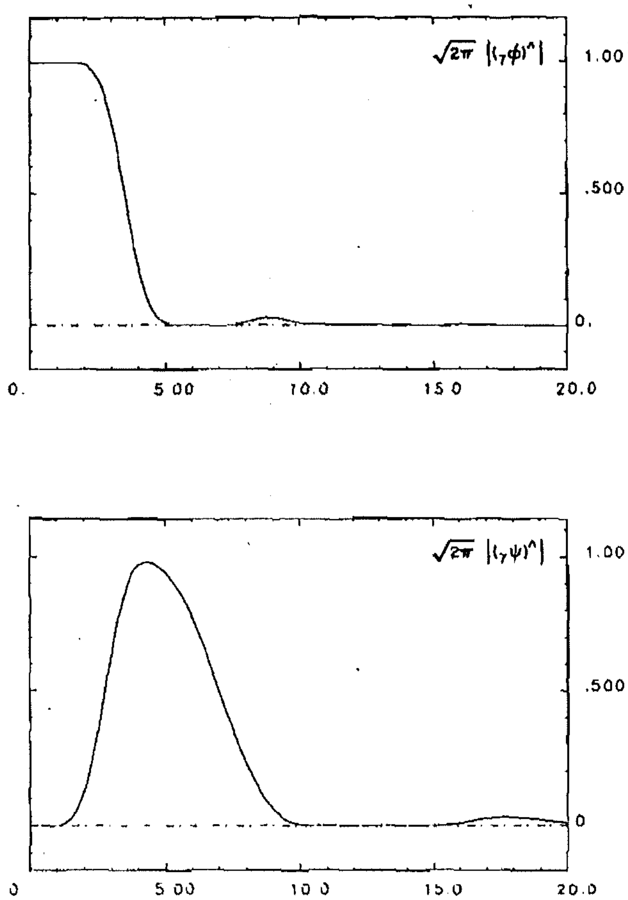

Figure 7. Continued

One easily checks that

$$J_{N,m;2k} = \sum_l a_{N,2l} J_{N,m-1;k-l},$$

(A.6)

$$J_{N,m;2k+1} = \sum_l a_{N,2l+1} J_{N,m-1;k-l},$$

with $J_{N,0;k} = a_{N,k}$, and where we implicitly make the assumption $a_{N,k} = 0$ for $|k| \geqq N$. The recursion (A.6) can be represented graphically, in a construction

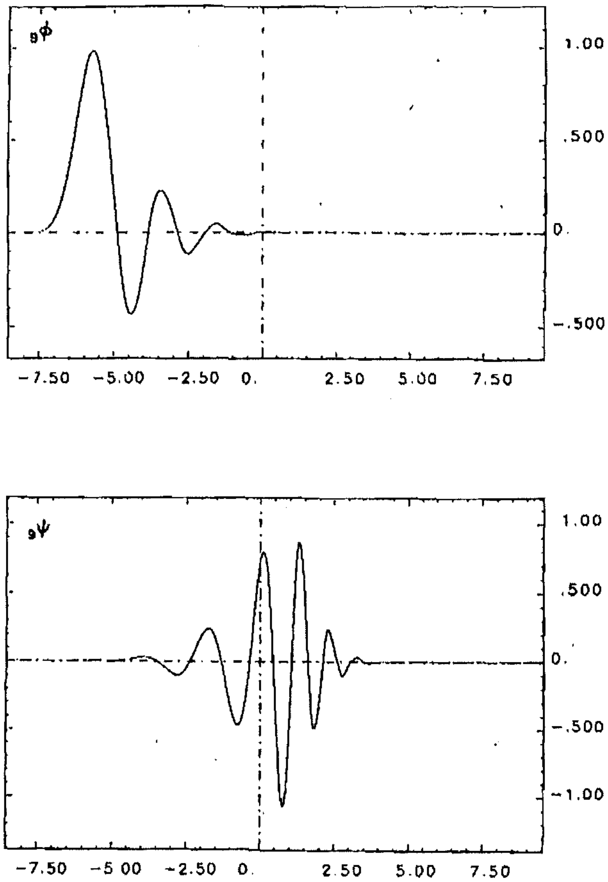

Figure 7. Continued

analogous to Figure 1. At level 0 we start with $J_{N,0}$; each successive $J_{N,n}$ is calculated from $J_{N,n-1}$ by a tree algorithm (see Figure 8). To evaluate the left-hand side of (A.2) we need to compute

$$\text{(A.7)} \qquad \int_0^{2\pi} d\xi \prod_{j=0}^{m} P_N\left(\sin^2(2^{j-1}\xi)\right) = J_{N,m;0}.$$

One can check directly from the recursion (A.6), or one can verify on the graphical representation (see Figure 8b) that only the $J_{N,m';l}$, $0 \leqq m' < m$, with $|l| \leqq N-2$ play a role in the computation of $J_{N,m;0}$. Define $d_N = 2N - 3$. Then

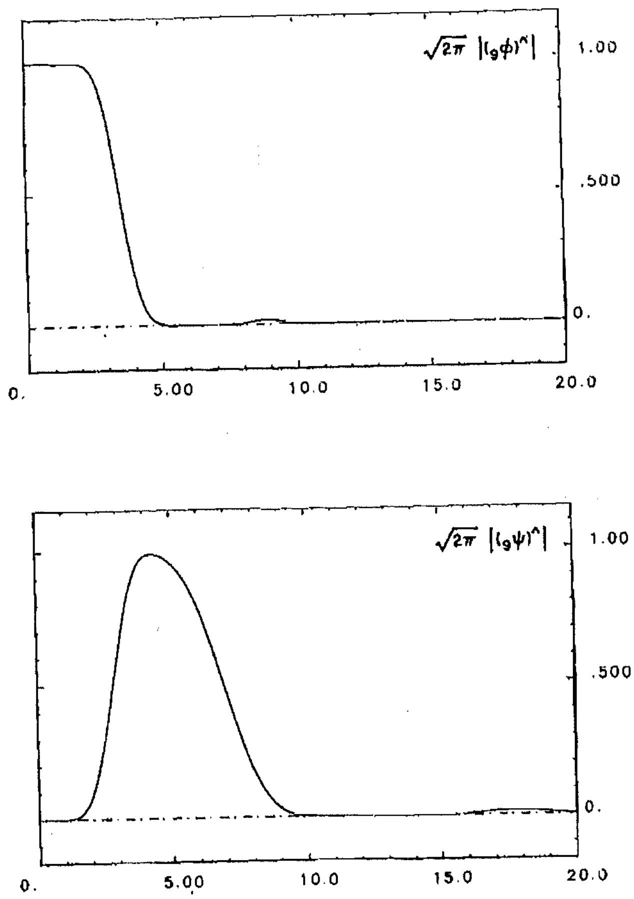

Figure 7. Continued

the set of relevant $J_{N,m';l}, \cdots, |l| \leq N-2, \cdots,$ define a vector $j_{N,m'}$ in $\mathbf{R}^{d_N}$,

(A.8) $$\left(j_{N,m'}\right)_k = J_{N,m';k}.$$

Note that d_N is always odd, $d_N = 2m_N + 1$, and that we index vectors $v \in \mathbf{R}^{d_N}$ by $j = -m_N, -m_n + 1, \cdots, 0, \cdots, m_N$ (see (A.8).) The recursion (A.6) defines a matrix T_N such that, for all m,

(A.9) $$j_{N,m+1} = T_N j_{N,m}.$$

Figure 8.
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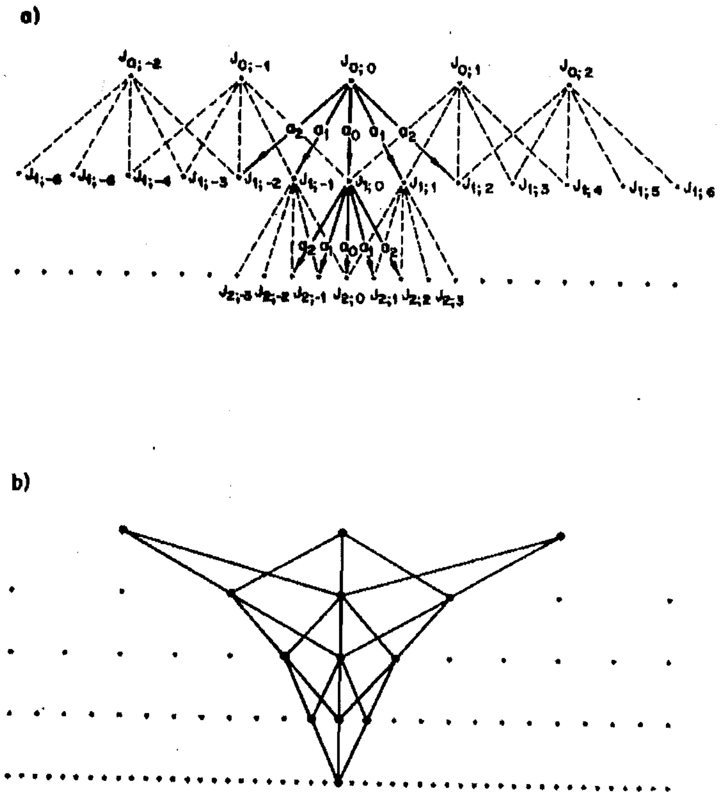

Figure 8. a. The tree algorithm for the construction of the $J_{N,m}$. For the sake of simplicity, we have taken $N = 3$. The index N is dropped on the figure.

b. Although the number of non-zero $J_{3,m;k}$ more than doubles (see a)) at every step, only 3 points, at *any* level, ultimately contribute to $J_{3,m;0}$. These are the points which can be reached from 0 by the tree, starting from the bottom.

This matrix has the following form, for N even:

(A.10)

$$T = \begin{pmatrix} a_{N-2} & a_{N-4} & a_{N-6} & \cdots & a_2 & a_0 & a_2 & \cdots & a_{N-2} & 0 & 0 & \cdots & 0 \\ a_{N-1} & a_{N-3} & a_{N-5} & \cdots & a_3 & a_1 & a_1 & \cdots & a_{N-3} & a_{N-1} & 0 & \cdots & 0 \\ 0 & a_{N-2} & a_{N-4} & \cdots & a_4 & a_2 & a_0 & \cdots & a_{N-4} & a_{N-2} & 0 & \cdots & 0 \\ 0 & a_{N-1} & a_{N-3} & \cdots & a_5 & a_3 & a_1 & \cdots & a_{N-5} & a_{N-3} & a_{N-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{N-3} & a_{N-5} & a_{N-7} & \cdots & a_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{N-2} & a_{N-4} & a_{N-6} & \cdots & a_{N-2} \end{pmatrix}.$$

A completely analogous matrix is obtained for N odd. From (A.8)–(A.9) we have

$$J_{N,m;0} = \left(T_N^m j_{N,0}\right)_0.$$

Hence

$$\begin{aligned}\limsup_{m\to\infty} m^{-1}\log\left(J_{N,m;0}\right) &\le \limsup_{m\to\infty} m^{-1}\log\left[\|(T_N)^m\|\cdot\|j_{N,0}\|\right]\\ &\le \limsup_{m\to\infty} \log\left[\|(T_N)^m\|^{1/m}\right] = \log\left(\rho(T_N)\right),\end{aligned}$$

where $\rho(T_N)$ is the spectral radius of T_N. In view of (A.7) it then follows that (A.1) is convergent, i.e., ${}_N\phi \in C^\alpha$, if $\alpha < N - 1 - \frac{1}{2}\log_2[\rho(T_N)]$. It suffices therefore to compute $\rho(T_N)$, which can be done numerically, provided N is not too large. Note that the problem can be reduced considerably by using the fact that T_N commutes with the involution I,

$$I_{ij} = \delta_{i,-j}$$

(where, as before, $i, j = -m_N, \cdots, 0, \cdots, m_N$). This effectively reduces the problem of a $d_N \times d_N$ matrix to a $(m_N + 1) \times (m_N + 1)$ matrix.

If $N = 2$, then $d_N = 1$, and the matrix T_1 is given by a single number, $T_1 = a_{1;0} = 2$. Therefore one finds ${}_1\phi \in C^\alpha$ if $\alpha < \frac{1}{2}$. The cause of this simplification can be understood by looking at Figure 8b. For $N = 2$, the "tree" reduces to a single vertical line: only one possible path leads from $J_{N,0;0}$ to $J_{N,n;0}$ if $N = 2$. This is equivalent to saying that in the product $\prod_{j=0}^{N} P_N(\sin^2(2^{j-1}\xi))$ only one possible combination of terms has frequency zero. This is the idea which was borrowed from Riesz's lemma (see Remark 4 following Proposition 4.7).

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